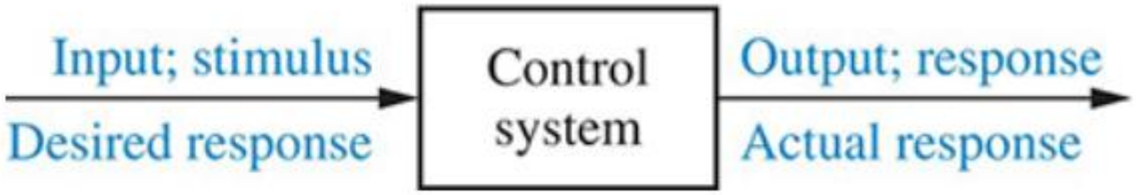
Control System

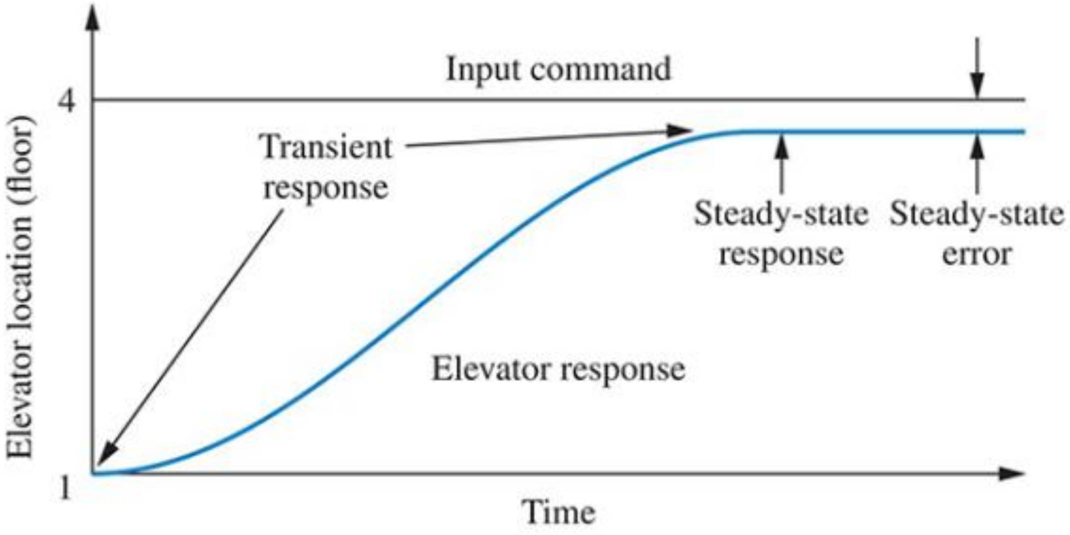
EEE 401 (Part 1)

Dept. of EEE, BUBT

Introduction



A control system consists of **subsystems** and **processes** (or plants) assembled for the purpose of obtaining a desired output with desired performance, given a specified input.

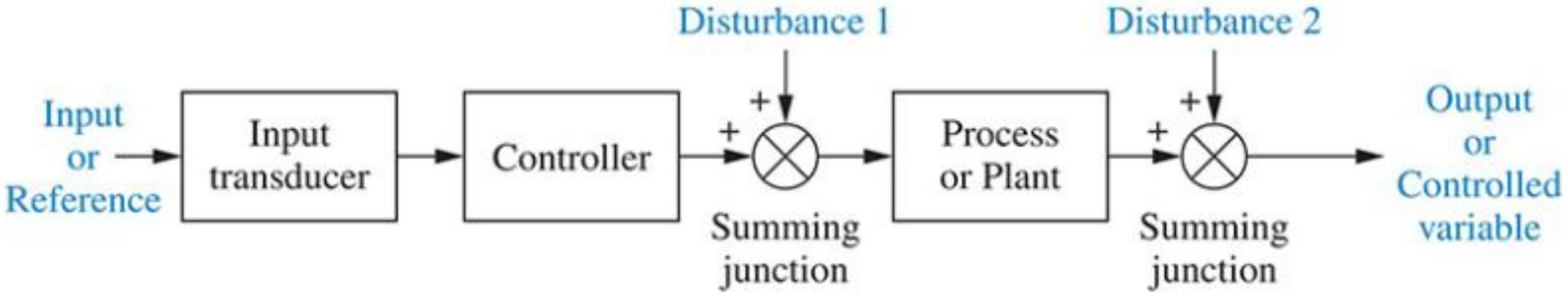


Why do we build control systems?

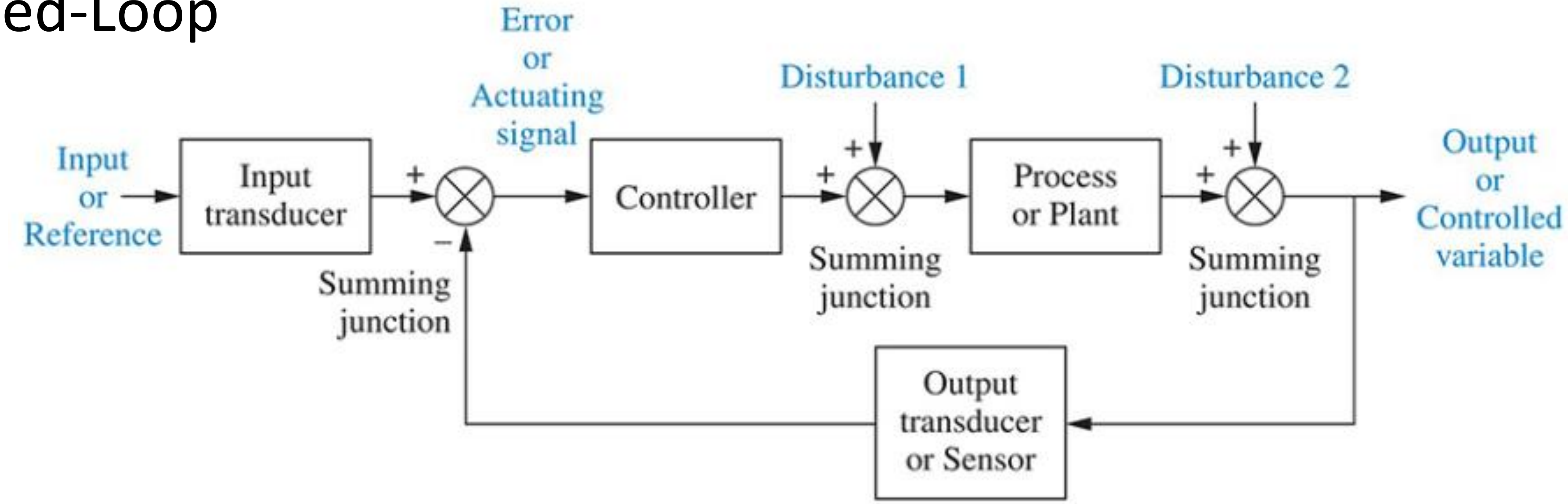
- Power amplification
  Example: rotation of radar antenna
- Remote control
  Example: robot in radioactive environment
- Convenience of input form
  Example: temperature control by thermostat
- Compensation for disturbances
  Example: cancellation of wind-force on antenna

System Configuration

- Open-Loop



- Closed-Loop



Laplace Transform Review

$$\mathscr{L}\left[f(t)\right] = F(s) = \int_{0-}^{\infty} f(t)\, e^{-st} dt$$

where $s = \sigma + j\omega$ (complex variable)

$$\mathscr{L}^{-1}\left[F(s)\right] = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F(s)\, e^{st} ds = f(t)\, u(t)$$

Laplace transform table

Item no.	$f(t)$	$F(s)$
1.	$\delta(t)$	1
2.	$u(t)$	$\frac{1}{s}$
3.	$tu(t)$	$\frac{1}{s^2}$
4.	$t^n u(t)$	$\frac{n!}{s^{n+1}}$
5.	$e^{-at}u(t)$	$\frac{1}{s+a}$
6.	$\sin \omega t u(t)$	$\frac{\omega}{s^2+\omega^2}$
7.	$\cos \omega t u(t)$	$\frac{s}{s^2+\omega^2}$

Laplace Transform Theorems

Item no.	Theorem	Name
1.	$\mathscr{L}\left[f(t)\right] = F(s) = \int_{0-}^{\infty} f(t)\, e^{-st} dt$	Definition
2.	$\mathscr{L}\left[kf(t)\right] = kF(s)$	Linearity theorem
3.	$\mathscr{L}\left[f_1(t) + f_2(t)\right] = F_1(s) + F_2(s)$	Linearity theorem
4.	$\mathscr{L}\left[e^{-at} f(t)\right] = F(s+a)$	Frequency shift theorem
5.	$\mathscr{L}\left[f(t-T)\right] = e^{-sT} F(s)$	Time shift theorem
6.	$\mathscr{L}\left[f(at)\right] = \frac{1}{a} F\left(\frac{s}{a}\right)$	Scaling theorem
7.	$\mathscr{L}\left[\frac{df}{dt}\right] = sF(s) - f(0-)$	Differentiation theorem
8.	$\mathscr{L}\left[\frac{d^2 f}{dt^2}\right] = s^2 F(s) - sf(0-) - f(0-)$	Differentiation theorem
9.	$\mathscr{L}\left[\frac{d^n f}{dt^n}\right] = s^n F(s) - \sum_{k=1}^{n} s^{n-k} f^{k-1}(0-)$	Differentiation theorem
10.	$\mathscr{L}\left[\int_{0-}^{t} f(\tau)\, d\tau\right] = \frac{F(s)}{s}$	Integration theorem

Example

Find the Laplace transform of $f(t) = Ae^{-at}u(t)$.

$$F(s) = \int_0^\infty f(t)\, e^{-st}\ dt = \int_0^\infty Ae^{-at}e^{-st}\ dt \ = \ A\int_0^\infty e^{-(s+a)t}\ dt$$

$$= \ -\frac{A}{s+a}e^{-(s+a)t}\Big|_{t=0}^{\infty} = \frac{A}{s+a}$$

Find the inverse Laplace transform of $F_1(s) = 1/(s + 3)^2$.

inverse transform of $F(s) = 1/s^2$ is $tu(t)$

inverse transform of $F(s + a) = 1/(s + a)^2$ is $e^{-at}tu(t)$

Hence, $f_1(t) = e^{-3t}tu(t)$.

Find the Laplace transform of $f(t) = te^{-5t}$

$$F(s) = \frac{1}{(s+5)^2}$$

continued…

Given the following differential equation, solve for $y(t)$ if all initial conditions are zero. Use Laplace transform.

$$\frac{d^2y}{dt^2} + 12\frac{dy}{dt} + 32y = 32u(t)$$

initial conditions of $y(t)$ and $dy(t)/dt$ given by $y(0-) = 0$ and $\dot{y}(0-) = 0$

$$s^2Y(s) + 12sY(s) + 32Y(s) = \frac{32}{s}$$

$$Y(s) = \frac{32}{s(s^2+12s+32)} = \frac{32}{s(s+4)(s+8)}$$

$$Y(s) = \frac{32}{s(s+4)(s+8)} = \frac{K_1}{s} + \frac{K_2}{(s+4)} + \frac{K_3}{(s+8)}$$

$$Y(s) = \frac{1}{s} - \frac{2}{(s+4)} + \frac{1}{(s+8)}$$

$$K_1 = \left.\frac{32}{(s+4)(s+8)}\right|_{s\to 0} = 1$$

$$K_2 = \left.\frac{32}{s(s+8)}\right|_{s\to -4} = -2$$

$$K_3 = \left.\frac{32}{s(s+4)}\right|_{s\to -8} = 1$$

continued...

$$Y(s) = \frac{1}{s} - \frac{2}{(s+4)} + \frac{1}{(s+8)}$$

$$y(t) = \left(1 - 2e^{-4t} + e^{-8t}\right) u(t)$$

$$F(s) = \frac{2}{(s+1)(s+2)^2}$$

$$F(s) = \frac{2}{(s+1)(s+2)^2} = \frac{K_1}{(s+1)} + \frac{K_2}{(s+2)^2} + \frac{K_3}{(s+2)}$$

$$f(t) = 2e^{-t} - 2te^{-2t} - 2e^{-2t}$$

$$K_1 = 2$$
$$K_2 = -2$$
$$K_3 = -2$$

continued...

Find the inverse Laplace transform of $F(s) = 10/[s(s+2)(s+3)^2]$

$$F(s) = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{(s+3)^2} + \frac{D}{(s+3)}$$

$$f(t) = \frac{5}{9} - 5e^{-2t} + \frac{10}{3}te^{-3t} + \frac{40}{9}e^{-3t}$$

$$A = \left.\frac{10}{(s+2)(s+3)^2}\right|_{S\to 0} = \frac{5}{9}$$

$$B = \left.\frac{10}{s(s+3)^2}\right|_{S\to -2} = -5$$

$$C = \left.\frac{10}{s(s+2)}\right|_{S\to -3} = \frac{10}{3}$$

$$D = \frac{40}{9}$$

continued...

$$F(s) = \frac{3}{s(s^2+2s+5)}$$

$$\frac{3}{s(s^2+2s+5)} = \frac{K_1}{s} + \frac{K_2 s + K_3}{s^2+2s+5}$$

$$F(s) = \frac{3}{s(s^2+2s+5)} = \frac{3/5}{s} - \frac{3}{5}\frac{s+2}{s^2+2s+5}$$

$K_1 = \frac{3}{5}$ $K_2 = -\frac{3}{5}$ $K_3 = -\frac{6}{5}$

$$\mathscr{L}\left[Ae^{-at}\cos\omega t\right] = \frac{A(s+a)}{(s+a)^2+\omega^2}$$

$$\mathscr{L}\left[Be^{-at}\sin\omega t\right] = \frac{B\omega}{(s+a)^2+\omega^2}$$

$$\mathscr{L}\left[Ae^{-at}\cos\omega t + Be^{-at}\sin\omega t\right] = \frac{A(s+a)+B\omega}{(s+a)^2+\omega^2}$$

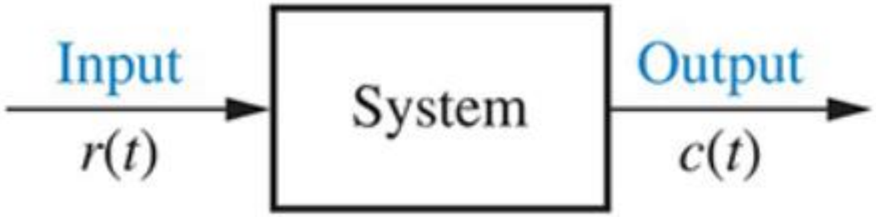
$$F(s) = \frac{3/5}{s} - \frac{3}{5}\frac{(s+1)+(1/2)(2)}{(s+1)^2+2^2}$$

$$\rightarrow$$

$$f(t) = \frac{3}{5} - \frac{3}{5}e^{-t}\left(\cos 2t + \frac{1}{2}\sin 2t\right)$$

Transfer Function

$$a_n \frac{d^n c(t)}{dt^n} + a_{n-1} \frac{d^{n-1} c(t)}{dt^{n-1}} + \cdots + a_0 c(t) = b_m \frac{d^m r(t)}{dt^m} + b_{m-1} \frac{d^{m-1} r(t)}{dt^{m-1}} + \cdots + b_0 r(t)$$



Note: The input, $r(t)$, stands for *reference input*.
The output, $c(t)$, stands for *controlled variable*.

$$a_n s^n C(s) + a_{n-1} s^{n-1} C(s) + \cdots + a_0 C(s) + \text{initial condition terms involving } c(t)$$
$$= b_m s^m R(s) + b_{m-1} s^{m-1} R(s) + \cdots + b_0 R(s) + \text{initial condition terms involving } r(t)$$

assume that *all initial conditions are zero*

$$(a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0) C(s) = (b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0) R(s)$$

$$\frac{C(s)}{R(s)} = G(s) = \frac{(b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0)}{(a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0)}$$

$$C(s) = R(s) G(s)$$

Example

Find the transfer function represented by

$$\frac{dc(t)}{dt} + 2c(t) = r(t)$$

find the response, $c(t)$ to an input, $r(t) = u(t)$ assuming zero initial conditions.

$$sC(s) + 2C(s) = R(s)$$

$$G(s) = \frac{C(s)}{R(s)} = \frac{1}{s+2}$$

$$R(s) = 1/s$$

$$C(s) = R(s)G(s) = \frac{1}{s(s+2)}$$

$$C(s) = \frac{1/2}{s} - \frac{1/2}{s+2} \quad \rightarrow \quad c(t) = \frac{1}{2} - \frac{1}{2}e^{-2t}$$

continued…

Find the transfer function, $G(s) = C(s)/R(s)$

$$\frac{d^3c}{dt^3} + 3\frac{d^2c}{dt^2} + 7\frac{dc}{dt} + 5c = \frac{d^2r}{dt^2} + 4\frac{dr}{dt} + 3r.$$

$$s^3C(s) + 3s^2C(s) + 7sC(s) + 5C(s) = s^2R(s) + 4sR(s) + 3R(s)$$

$$(s^3 + 3s^2 + 7s + 5)C(s) = (s^2 + 4s + 3)R(s)$$

$$G(s) = \frac{C(s)}{R(s)} = \frac{s^2 + 4s + 3}{s^3 + 3s^2 + 7s + 5}$$

Find the differential equation corresponding to the transfer function

$$G(s) = \frac{2s + 1}{s^2 + 6s + 2}$$

$$G(s) = \frac{C(s)}{R(s)} = \frac{2s + 1}{s^2 + 6s + 2} \rightarrow \frac{d^2c}{dt^2} + 6\frac{dc}{dt} + 2c = 2\frac{dr}{dt} + r$$

continued…

Find the ramp response for a system whose transfer function is

$$G(s) = \frac{s}{(s+4)(s+8)}$$

$$C(s) = R(s)G(s) = \frac{1}{s^2}\frac{s}{(s+4)(s+8)} = \frac{1}{s(s+4)(s+8)}$$

$$\frac{1}{s(s+4)(s+8)} = \frac{A}{s} + \frac{B}{(s+4)} + \frac{C}{(s+8)}$$

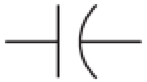
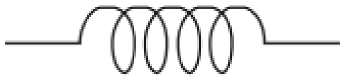
$$c(t) = \frac{1}{32} - \frac{1}{16}e^{-4t} + \frac{1}{32}e^{-8t}$$

$$A = \left.\frac{1}{(s+4)(s+8)}\right|_{S\to 0} = \frac{1}{32}$$

$$B = \left.\frac{1}{s(s+8)}\right|_{S\to -4} = -\frac{1}{16}$$

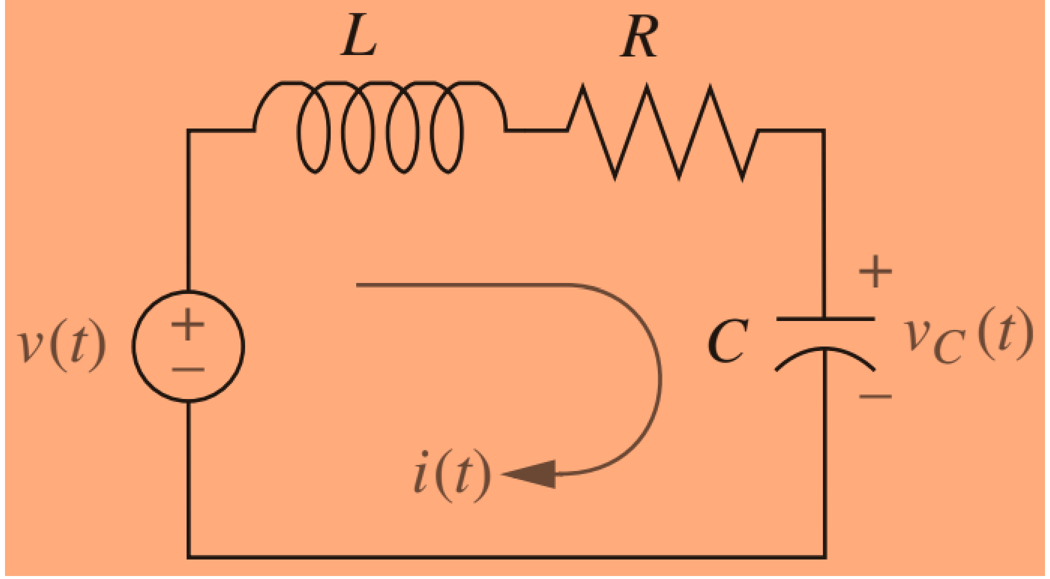
$$C = \left.\frac{1}{s(s+4)}\right|_{S\to -8} = \frac{1}{32}$$

Electrical Network Transfer Functions

Component	Voltage-current	Current-voltage	Voltage-charge	Impedance $Z(s) = V(s)/I(s)$	Admittance $Y(s) = I(s)/V(s)$
 Capacitor	$v(t) = \frac{1}{C}\int_0^t i(\tau)d\tau$	$i(t) = C\frac{dv(t)}{dt}$	$v(t) = \frac{1}{C}q(t)$	$\frac{1}{Cs}$	Cs
 Resistor	$v(t) = Ri(t)$	$i(t) = \frac{1}{R}v(t)$	$v(t) = R\frac{dq(t)}{dt}$	R	$\frac{1}{R} = G$
 Inductor	$v(t) = L\frac{di(t)}{dt}$	$i(t) = \frac{1}{L}\int_0^t v(\tau)d\tau$	$v(t) = L\frac{d^2q(t)}{dt^2}$	Ls	$\frac{1}{Ls}$

Example

Find the transfer function relating the capacitor voltage, $V_C(s)$, to the input voltage, $V(s)$



$$L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int_0^t i(\tau)\,d\tau = v(t)$$

$$i(t) = dq(t)/dt$$

$$L\frac{d^2q(t)}{dt^2} + R\frac{dq(t)}{dt} + \frac{1}{C}q(t) = v(t)$$

$$q(t) = Cv_C(t)$$

$$LC\frac{d^2v_C(t)}{dt^2} + RC\frac{dv_C(t)}{dt} + v_C(t) = v(t)$$

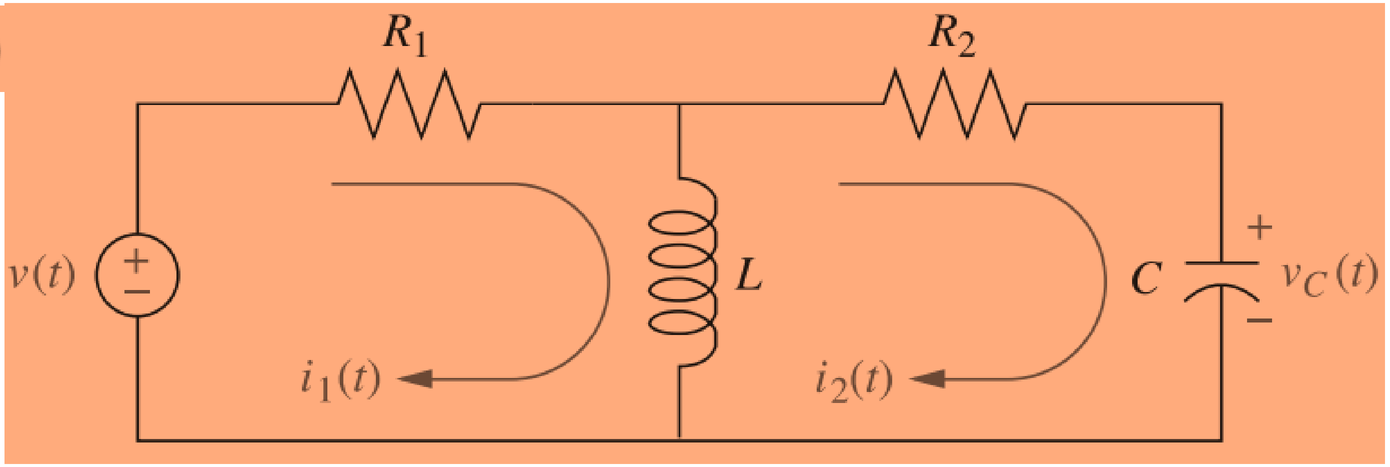
$$\left(LCs^2 + RCs + 1\right)V_C(s) = V(s)$$

assuming zero initial conditions

$$\frac{V_C(s)}{V(s)} = \frac{1/LC}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

continued…

find the transfer function, $I_2(s)/V(s)$

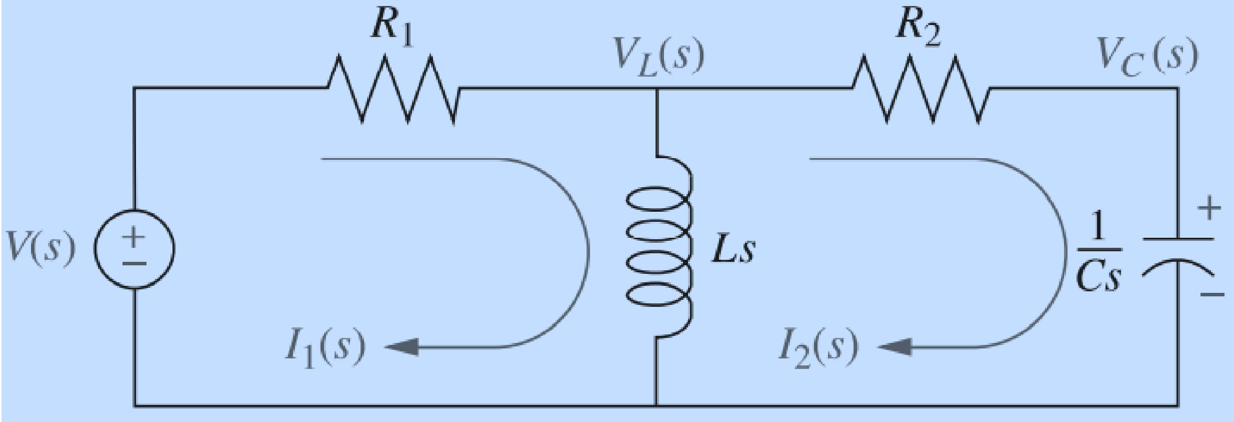


$$R_1 I_1(s) + LsI_1(s) - LsI_2(s) = V(s)$$

$$LsI_2(s) + R_2 I_2(s) + \frac{1}{Cs} I_2(s) - LsI_1(s) = 0$$

$$(R_1 + Ls)I_1(s) \qquad - LsI_2(s) = V(s)$$

$$-LsI_1(s) + \left(Ls + R_2 + \frac{1}{Cs}\right) I_2(s) = 0$$



$$\Delta = \begin{vmatrix} (R_1 + Ls) & -Ls \\ -Ls & \left(Ls + R_2 + \frac{1}{Cs}\right) \end{vmatrix}$$

continued...

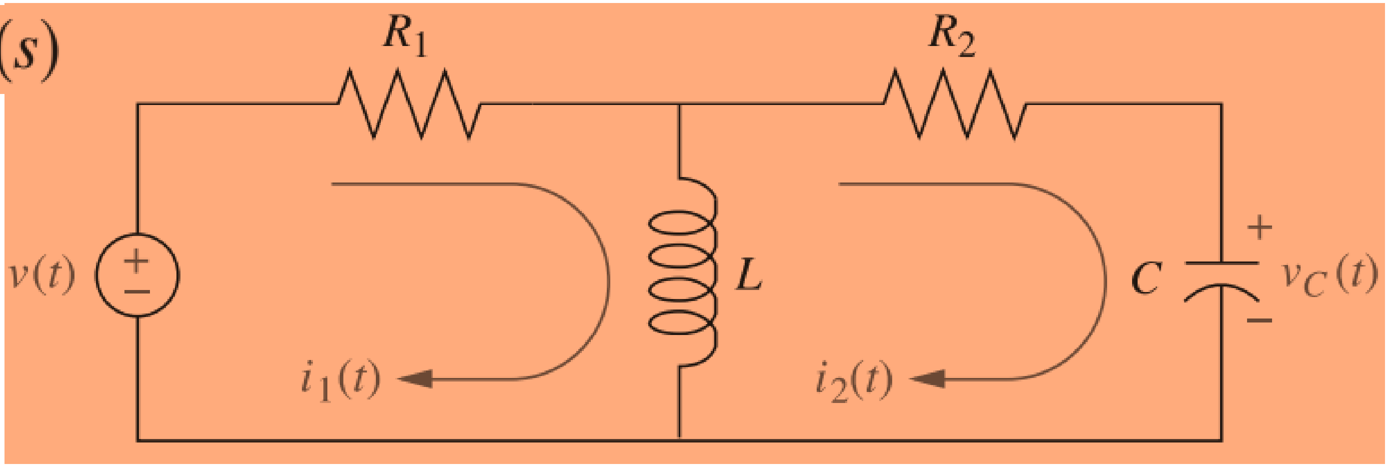
$$\Delta = \begin{vmatrix} (R_1 + Ls) & -Ls \\ -Ls & \left(Ls + R_2 + \dfrac{1}{Cs}\right) \end{vmatrix}$$

$$I_2(s) = \frac{\begin{vmatrix} (R_1 + Ls) & V(s) \\ -Ls & 0 \end{vmatrix}}{\Delta} = \frac{LsV(s)}{\Delta}$$

$$G(s) = \frac{I_2(s)}{V(s)} = \frac{Ls}{\Delta} = \frac{LCs^2}{(R_1 + R_2)LCs^2 + (R_1R_2C + L)s + R_1}$$

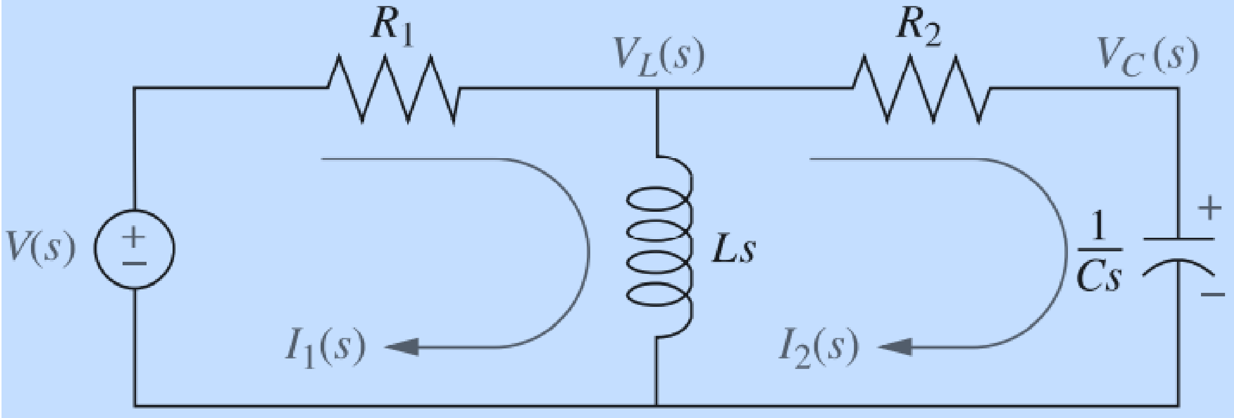
continued…

Find the transfer function, $V_C(s)/V(s)$



$$\frac{V_L(s) - V(s)}{R_1} + \frac{V_L(s)}{Ls} + \frac{V_L(s) - V_C(s)}{R_2} = 0$$

$$CsV_C(s) + \frac{V_C(s) - V_L(s)}{R_2} = 0$$



$$\left(G_1 + G_2 + \frac{1}{Ls}\right)V_L(s) \qquad - G_2 V_C(s) = V(s)G_1$$

$$-G_2 V_L(s) + (G_2 + Cs)V_C(s) = 0$$

continued…

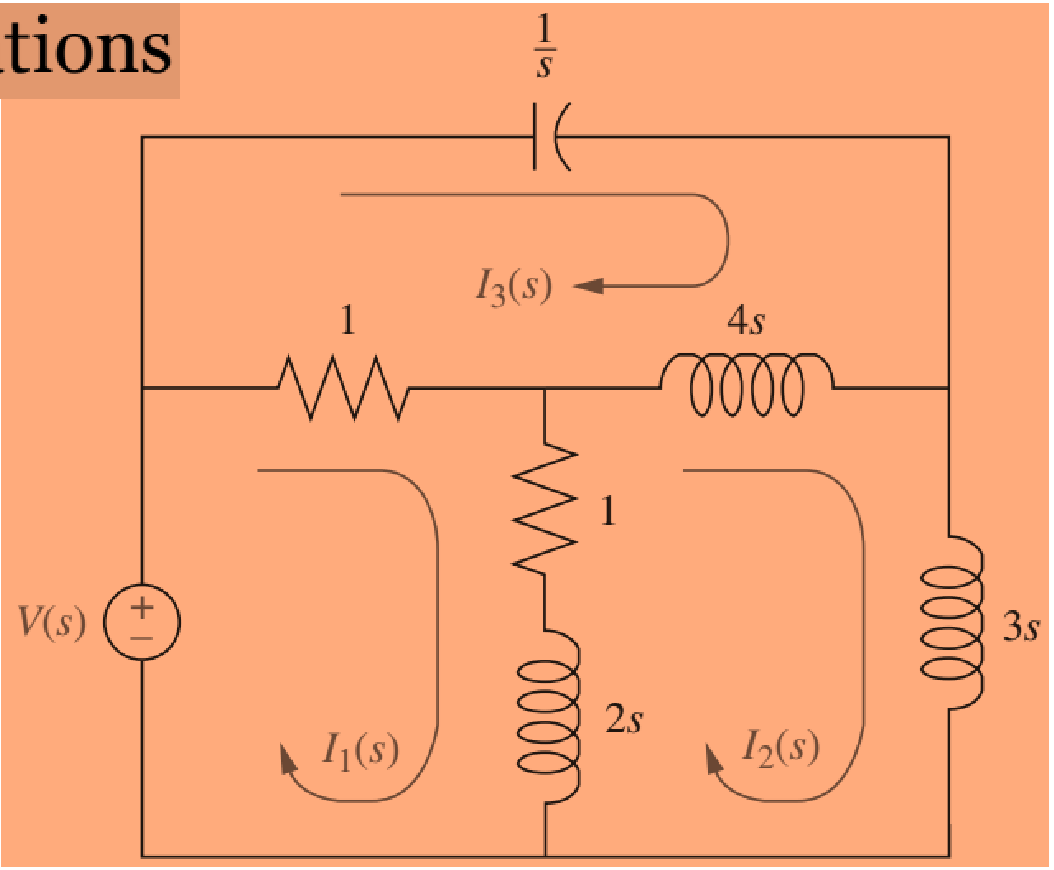
$$\frac{V_C(s)}{V(s)} = \frac{\frac{G_1G_2}{C}s}{(G_1+G_2)s^2 + \frac{G_1G_2L+C}{LC}s + \frac{G_2}{LC}}$$

Write, but do not solve, the mesh equations

$$+(2s+2)I_1(s) - (2s+1)I_2(s) - I_3(s) = V(s)$$

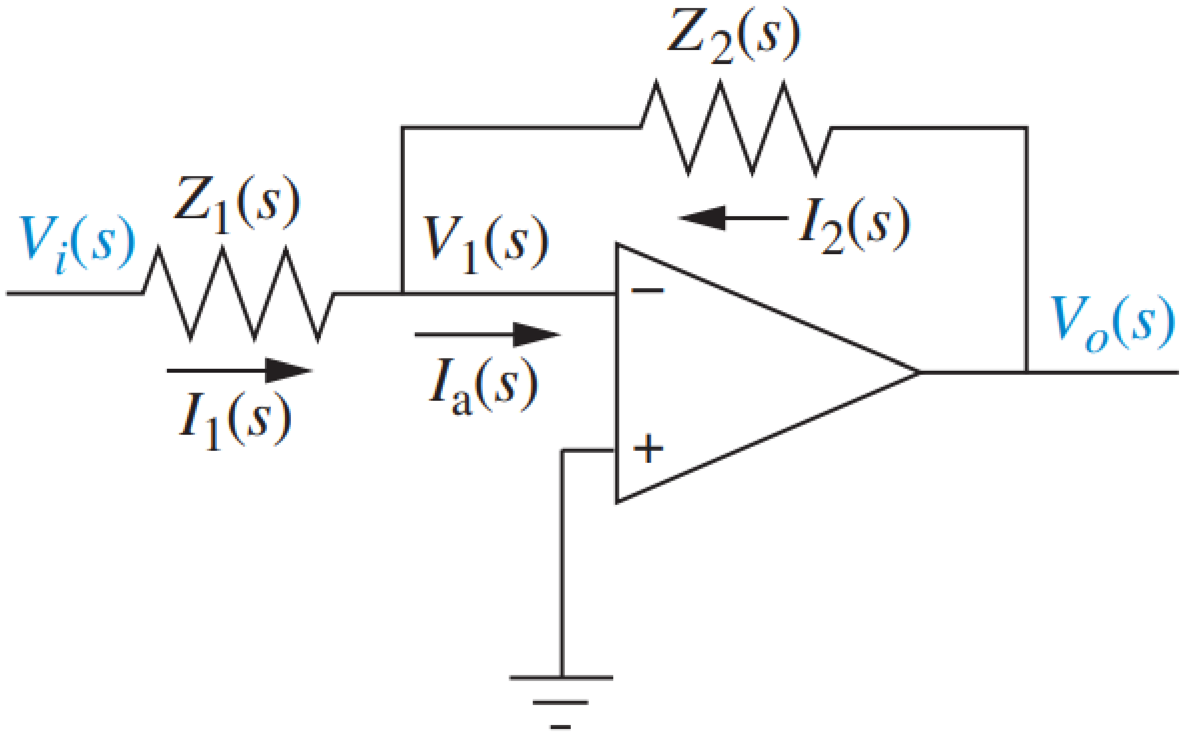
$$-(2s+1)I_1(s) + (9s+1)I_2(s) - 4sI_3(s) = 0$$

$$-I_1(s) - 4sI_2(s) + \left(4s+1+\frac{1}{s}\right)I_3(s) = 0$$

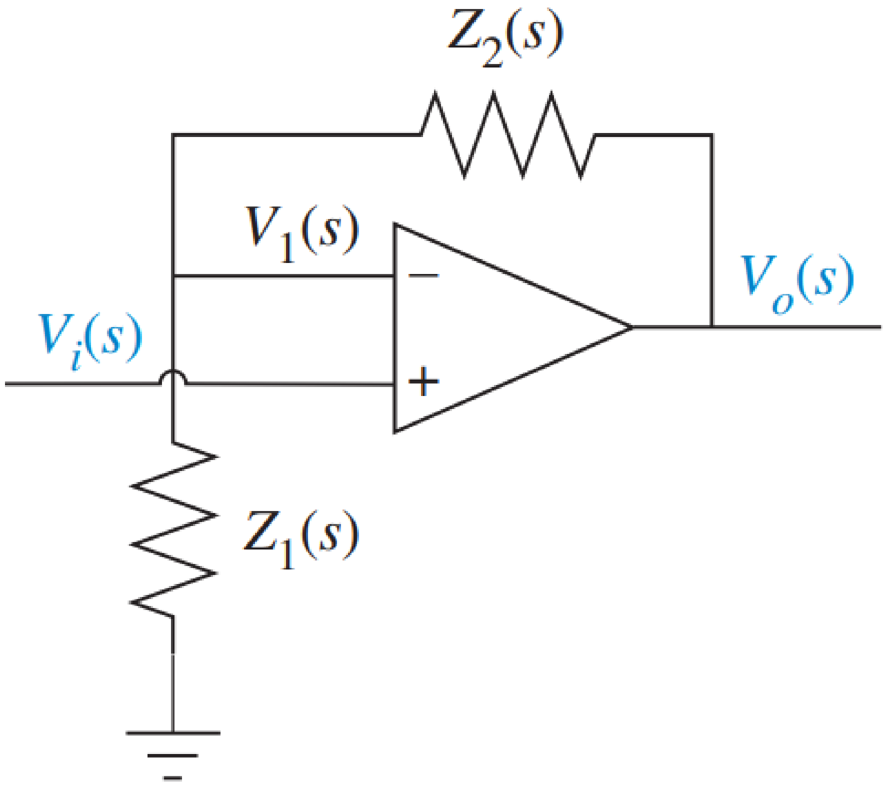


continued…

Op-Amp



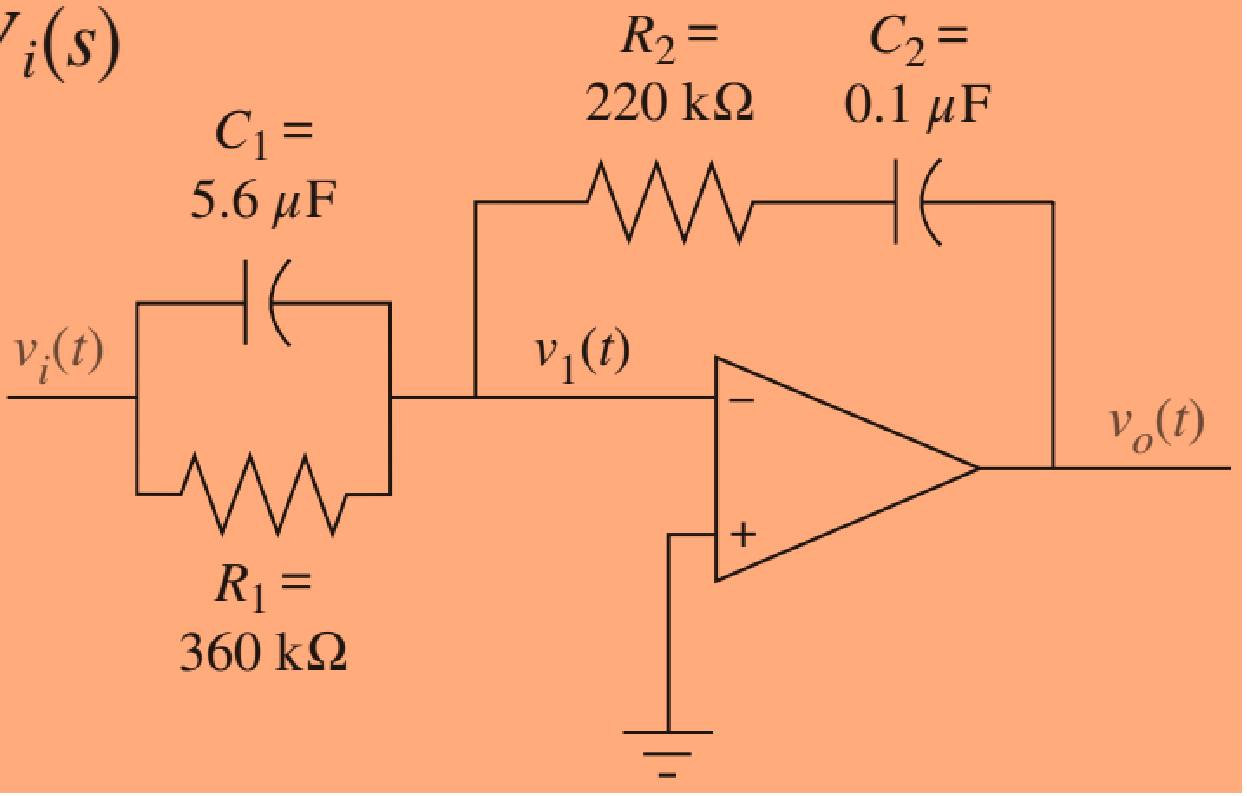
$$\frac{V_o(s)}{V_i(s)} = -\frac{Z_2(s)}{Z_1(s)}$$



$$\frac{V_o(s)}{V_i(s)} = \frac{Z_1(s) + Z_2(s)}{Z_1(s)}$$

continued…

Find the transfer function, $V_o(s)/V_i(s)$



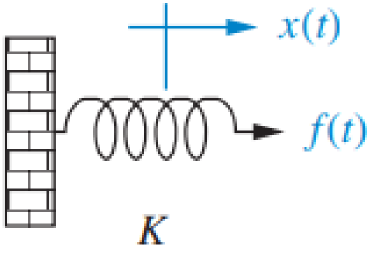
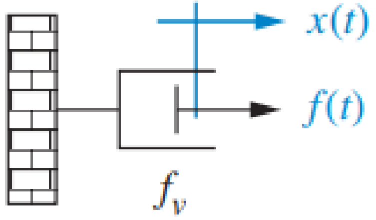
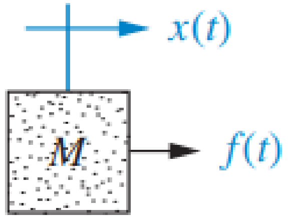
$$Z_1(s) = \frac{1}{C_1 s + \frac{1}{R_1}}$$

$$= \frac{1}{5.6 + 10^{-6}s + \frac{1}{360 \times 10^3}} = \frac{360 \times 10^3}{2.016s + 1}$$

$$Z_2(s) = R_2 + \frac{1}{C_2 s} = 220 \times 10^3 + \frac{10^7}{s}$$

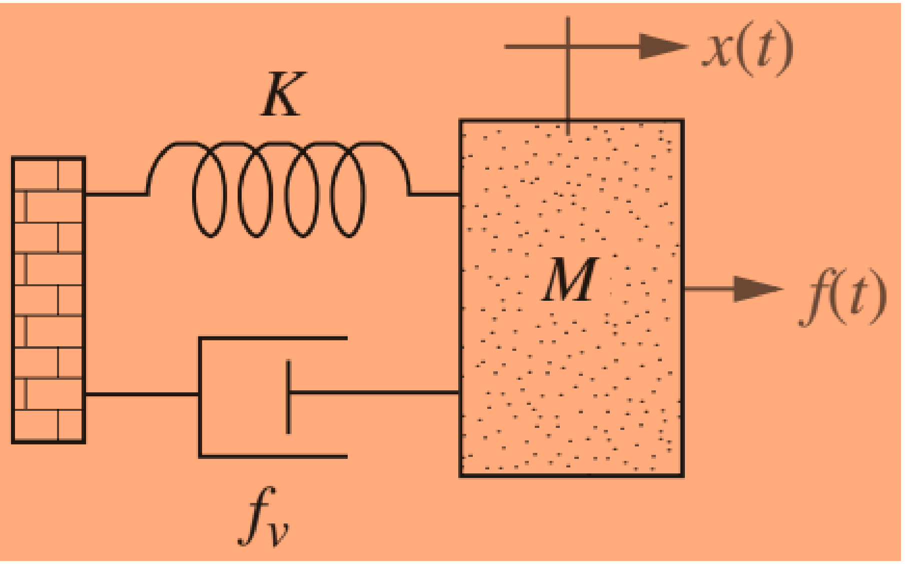
$$\frac{V_o(s)}{V_i(s)} = -1.232 \frac{s^2 + 45.95s + 22.55}{s}$$

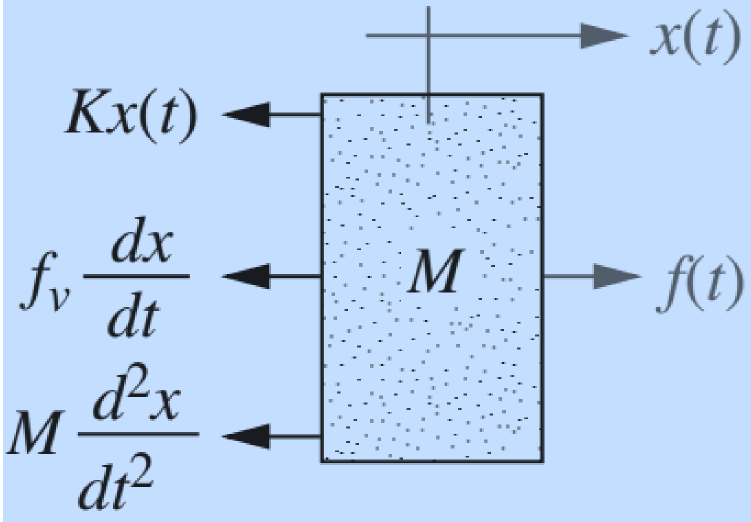
Translational Mechanical System

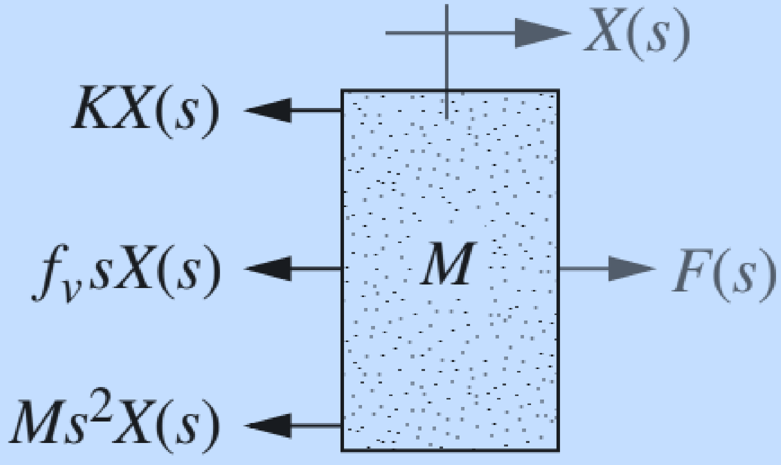
Component	Force-velocity	Force-displacement	Impedence $Z_M(s) = F(s)/X(s)$
Spring 	$f(t) = K\int_0^t v(\tau)d\tau$	$f(t) = Kx(t)$	K
Viscous damper 	$f(t) = f_v v(t)$	$f(t) = f_v \frac{dx(t)}{dt}$	$f_v s$
Mass 	$f(t) = M\frac{dv(t)}{dt}$	$f(t) = M\frac{d^2x(t)}{dt^2}$	Ms^2

Example

Find the transfer function, $X(s)/F(s)$







assuming zero initial conditions

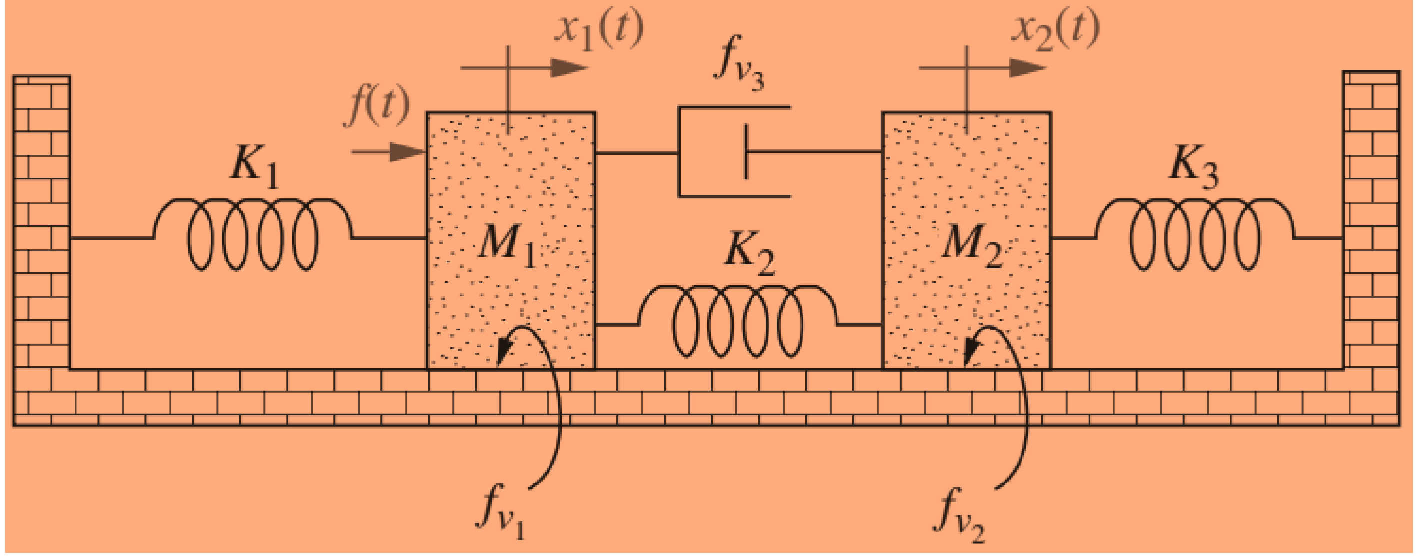
$$Ms^2X(s) + f_v sX(s) + KX(s) = F(s)$$

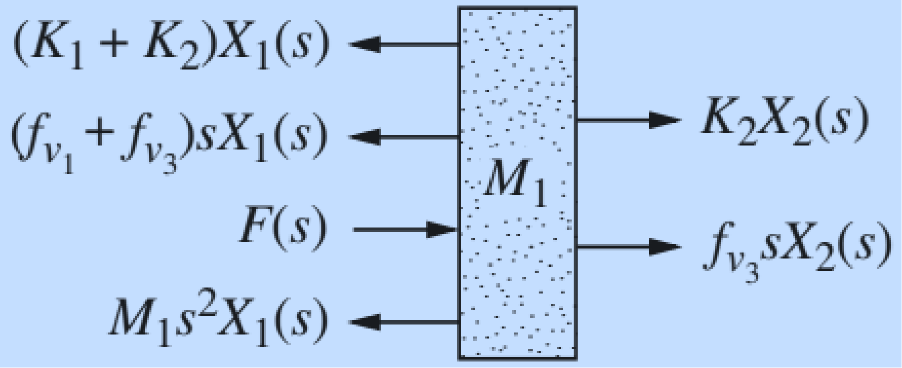
$$(Ms^2 + f_v s + K)X(s) = F(s)$$

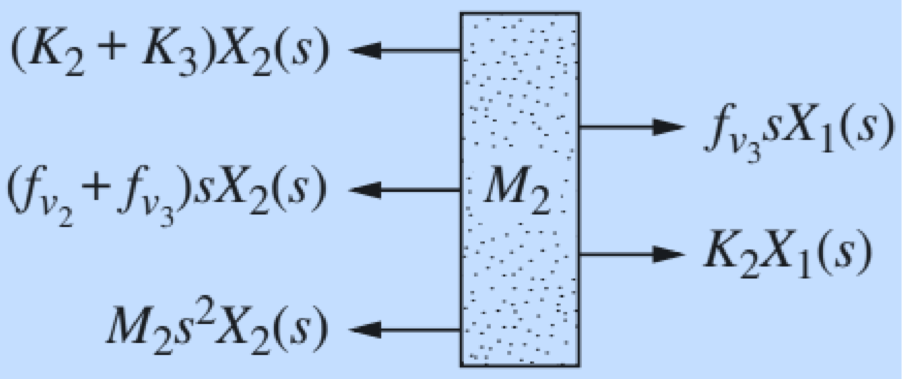
$$G(s) = \frac{X(s)}{F(s)} = \frac{1}{Ms^2 + f_v s + K}$$

continued...

Find the transfer function, $X_2(s)/F(s)$







$$[M_1 s^2 + (f_{v_1} + f_{v_3})s + (K_1 + K_2)]X_1(s) - (f_{v_3} s + K_2)X_2(s) = F(s)$$

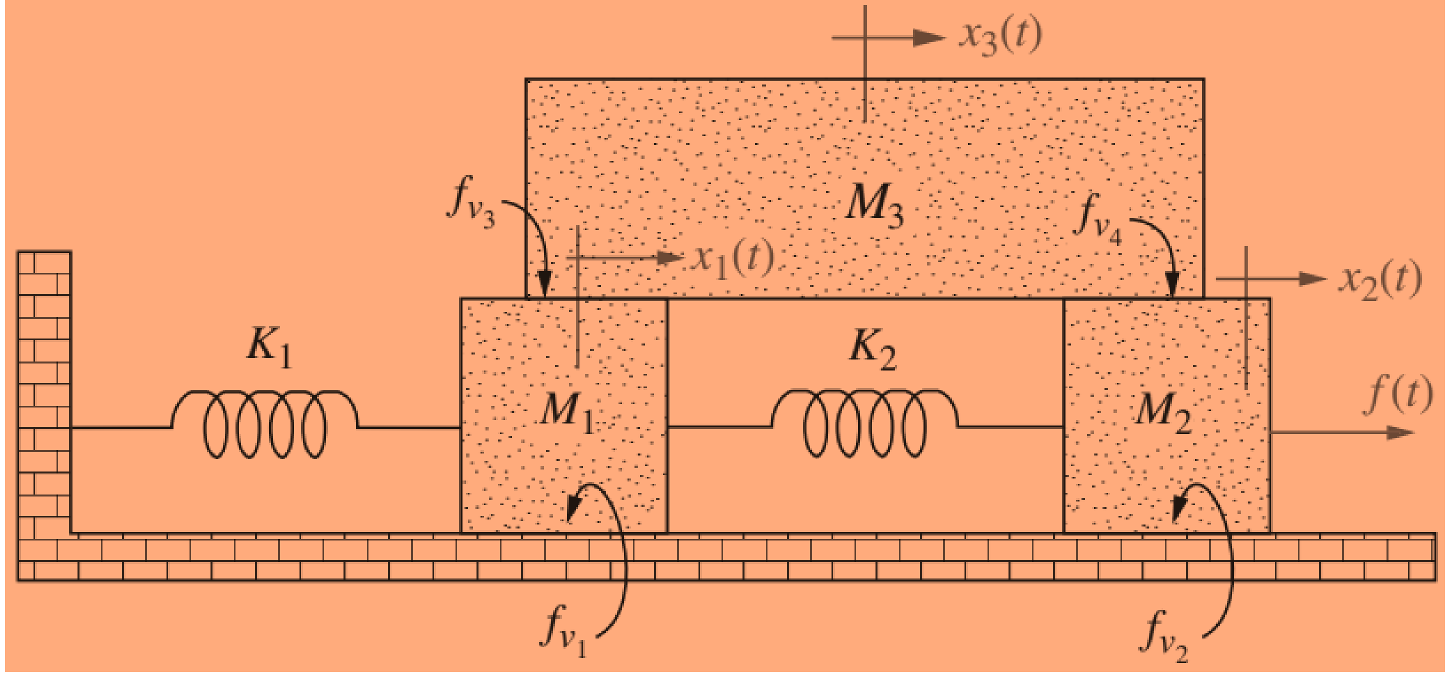
$$-(f_{v_3} s + K_2)X_1(s) + [M_2 s^2 + (f_{v_2} + f_{v_3})s + (K_2 + K_3)]X_2(s) = 0$$

$$\Delta = \begin{vmatrix} [M_1 s^2 + (f_{v_1} + f_{v_3})s + (K_1 + K_2)] & -(f_{v_3} s + K_2) \\ -(f_{v_3} s + K_2) & [M_2 s^2 + (f_{v_2} + f_{v_3})s + (K_2 + K_3)] \end{vmatrix}$$

$$\frac{X_2(s)}{F(s)} = G(s) = \frac{(f_{v_3} s + K_2)}{\Delta}$$

continued…

Write, but do not solve, the equations of motion for the mechanical network

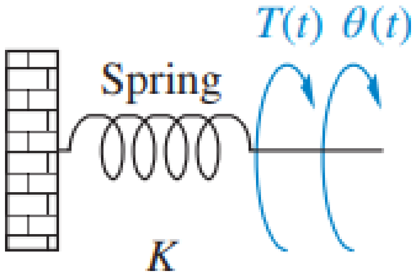
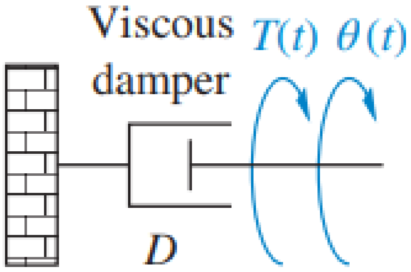
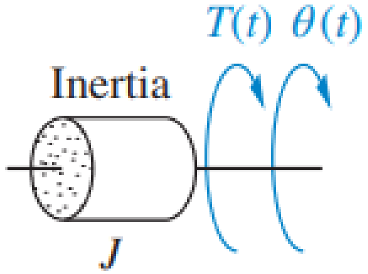


$$[M_1s^2 + (f_{v_1} + f_{v_3})s + (K_1 + K_2)]X_1(s) - K_2X_2(s) - f_{v_3}sX_3(s) = 0$$

$$-K_2X_1(s) + [M_2s^2 + (f_{v_2} + f_{v_4})s + K_2]X_2(s) - f_{v_4}sX_3(s) = F(s)$$

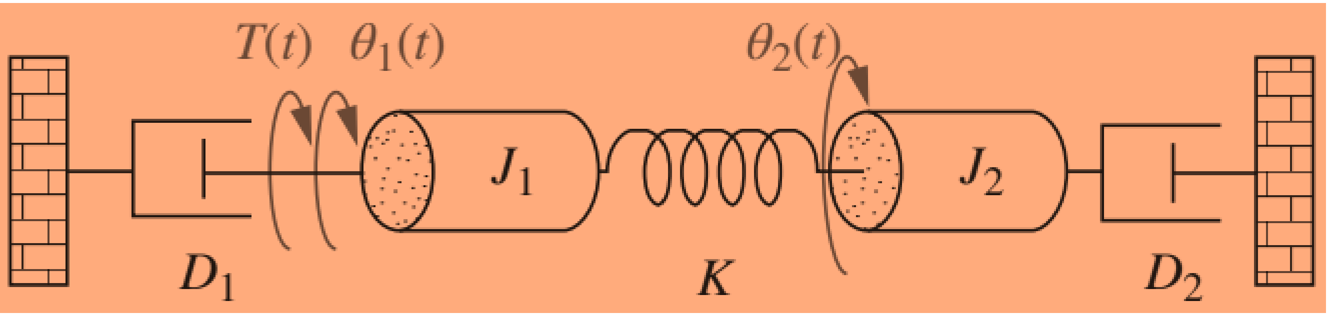
$$-f_{v_3}sX_1(s) - f_{v_4}sX_2(s) + [M_3s^2 + (f_{v_3} + f_{v_4})s]X_3(s) = 0$$

Rotational Mechanical System

Component	Torque-angular velocity	Torque-angular displacement	Impedence $Z_M(s) = T(s)/\theta(s)$
	$T(t) = K\int_0^t \omega(\tau)d\tau$	$T(t) = K\theta(t)$	K
	$T(t) = D\omega(t)$	$T(t) = D\frac{d\theta(t)}{dt}$	Ds
	$T(t) = J\frac{d\omega(t)}{dt}$	$T(t) = J\frac{d^2\theta(t)}{dt^2}$	Js^2

Example

Find the transfer function, $\theta_2(s)/T(s)$



$$(J_1s^2 + D_1s + K)\theta_1(s) \qquad - K\theta_2(s) = T(s)$$

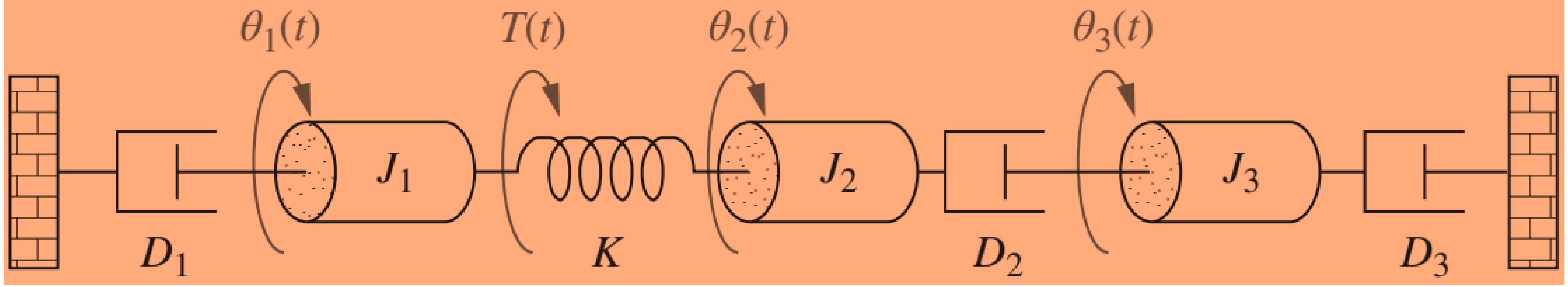
$$-K\theta_1(s) + (J_2s^2 + D_2s + K)\theta_2(s) = 0$$

$$\Delta = \begin{vmatrix} (J_1s^2 + D_1s + K) & -K \\ -K & (J_2s^2 + D_2s + K) \end{vmatrix}$$

$$\frac{\theta_2(s)}{T(s)} = \frac{K}{\Delta}$$

continued…

Write, but do not solve, the Laplace transform of the equations of motion

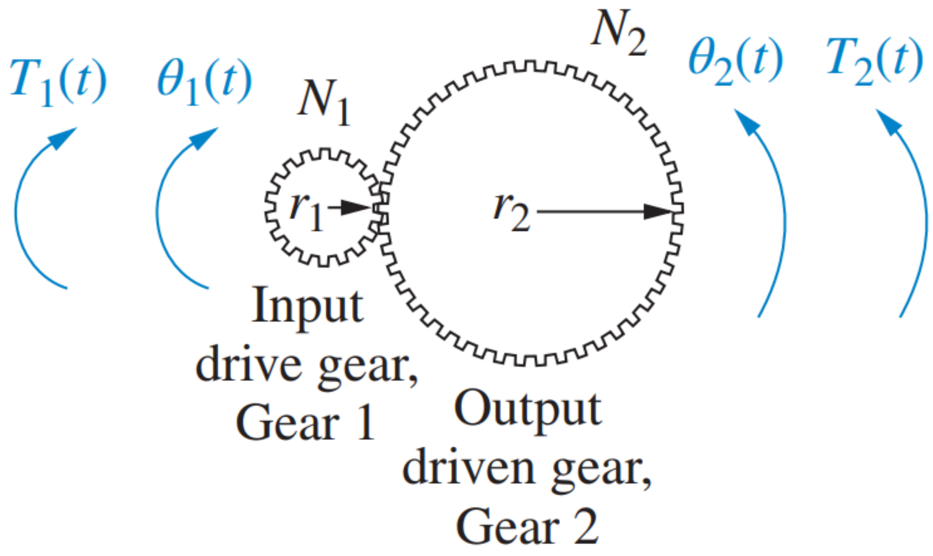


$$(J_1 s^2 + D_1 s + K)\theta_1(s) \quad -K\theta_2(s) \quad -0\theta_3(s) = T(s)$$

$$-K\theta_1(s) \quad +(J_2 s^2 + D_2 s + K)\theta_2(s) \quad -D_2 s\theta_3(s) = 0$$

$$-0\theta_1(s) \quad -D_2 s\theta_2(s) \quad +(J_3 s^2 + D_3 s + D_2 s)\theta_3(s) = 0$$

System with Gears



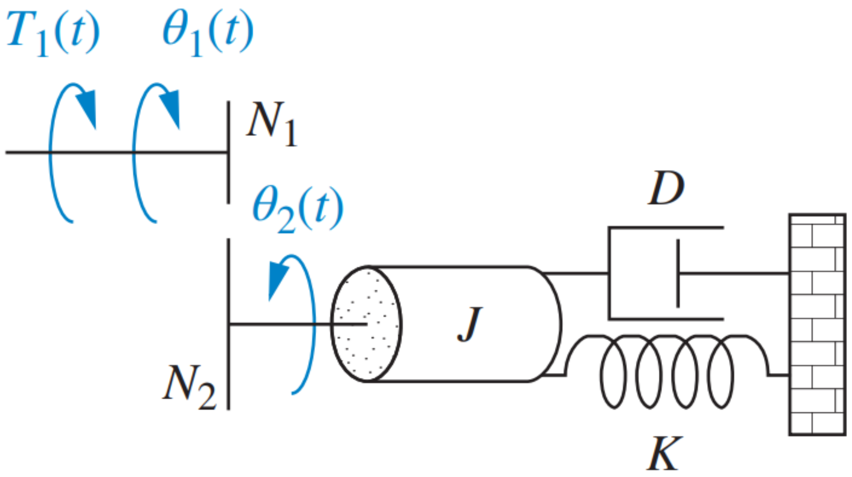
$$r_1\theta_1 = r_2\theta_2$$

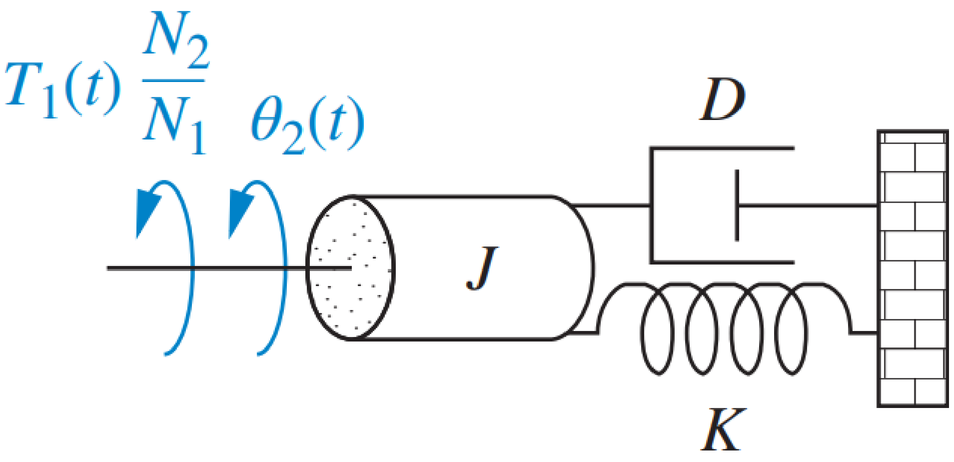
$$T_1\theta_1 = T_2\theta_2$$

$$\frac{\theta_2}{\theta_1} = \frac{r_1}{r_2} = \frac{N_1}{N_2}$$

$$\frac{T_2}{T_1} = \frac{\theta_1}{\theta_2} = \frac{N_2}{N_1}$$

continued...



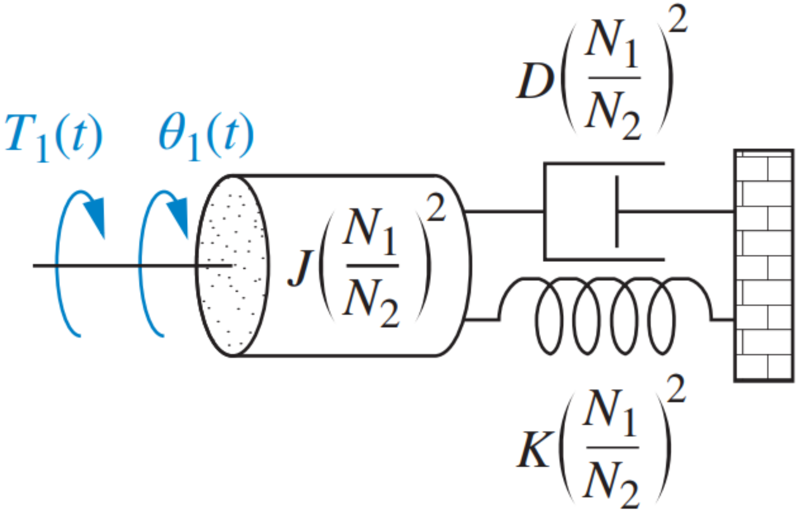


We want an equivalent system at θ_1 without gears.

$$(Js^2 + Ds + K)\,\theta_2(s) = T_1(s)\frac{N_2}{N_1}$$

$$(Js^2 + Ds + K)\frac{N_1}{N_2}\theta_1(s) = T_1(s)\frac{N_2}{N_1}$$

$$\left[J\left(\frac{N_1}{N_2}\right)^2 s^2 + D\left(\frac{N_1}{N_2}\right)^2 s + K\left(\frac{N_1}{N_2}\right)^2\right]\theta_1(s) = T_1(s)$$

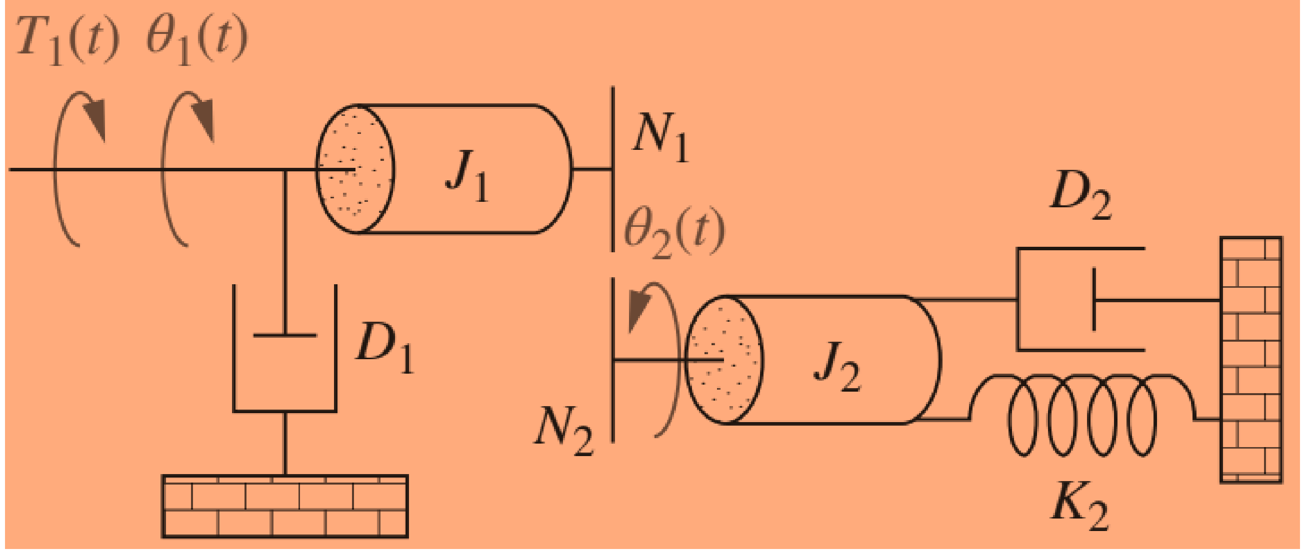


For impedance reflection, use

$$\left(\frac{\text{Number of teeth of gear on } \textit{destination} \text{ shaft}}{\text{Number of teeth of gear on } \textit{source} \text{ shaft}}\right)^2$$

Example

Find the transfer function, $\theta_2(s)/T_1(s)$



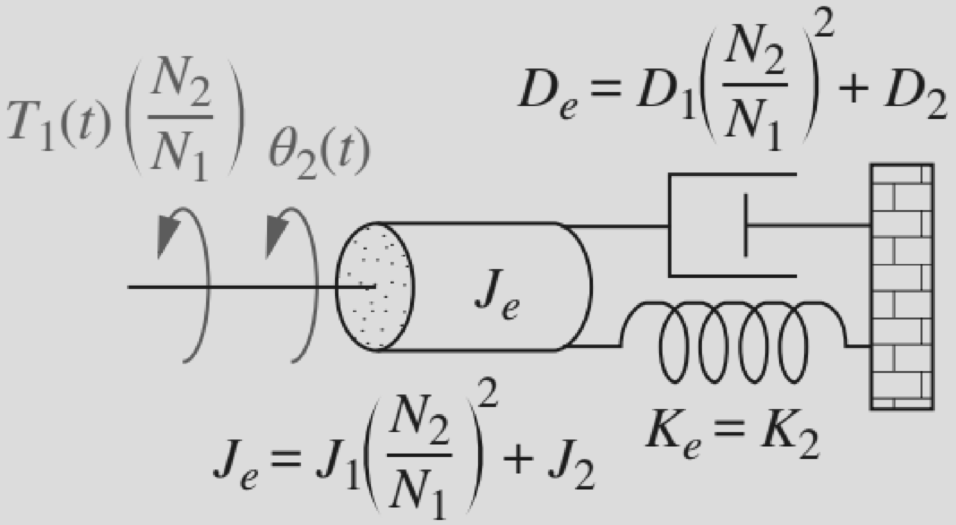
$$J_e = J_1\left(\frac{N_2}{N_1}\right)^2 + J_2$$

$$D_e = D_1\left(\frac{N_2}{N_1}\right)^2 + D_2$$

$$K_e = K_2$$

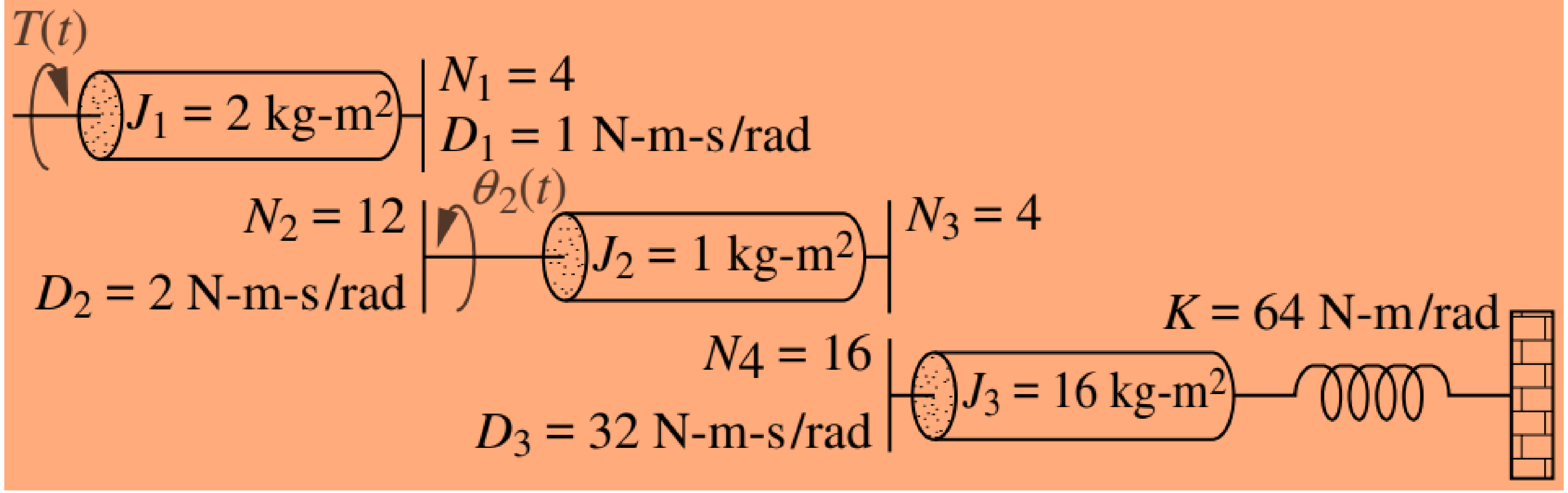
$$(J_e s^2 + D_e s + K_e)\theta_2(s) = T_1(s)\frac{N_2}{N_1}$$

$$G(s) = \frac{\theta_2(s)}{T_1(s)} = \frac{N_2/N_1}{J_e s^2 + D_e s + K_e}$$



continued…

Find $G(s) = \theta_2(s)/T(s)$



$$J_e = J_2 + J_1\left(\frac{N_2}{N_1}\right)^2 + J_3\left(\frac{N_3}{N_4}\right)^2$$
$$= 1 + 2\times\left(\frac{12}{4}\right)^2 + 16\times\left(\frac{4}{16}\right)^2 = 20$$

$$D_e = D_2 + D_1\left(\frac{N_2}{N_1}\right)^2 + D_3\left(\frac{N_3}{N_4}\right)^2$$
$$= 2 + 1\times\left(\frac{12}{4}\right)^2 + 32\times\left(\frac{4}{16}\right)^2 = 13$$

$$K_e = K\left(\frac{N_3}{N_4}\right)^2 = 64\times\left(\frac{4}{16}\right)^2 = 4$$

continued…

$$\therefore\ (J_e s^2 + D_e s + K_e)\theta_2(s) = T(s)\cdot\frac{N_2}{N_1}$$

$$\Longrightarrow (20s^2 + 13s + 4)\theta_2(s) = T(s)\cdot 3$$

$$\Longrightarrow \frac{\theta_2(s)}{T(s)} = \frac{3}{20s^2 + 13s + 4}$$

$$\Longrightarrow G(s) = \frac{3}{20s^2 + 13s + 4}$$

State-Space Representation

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

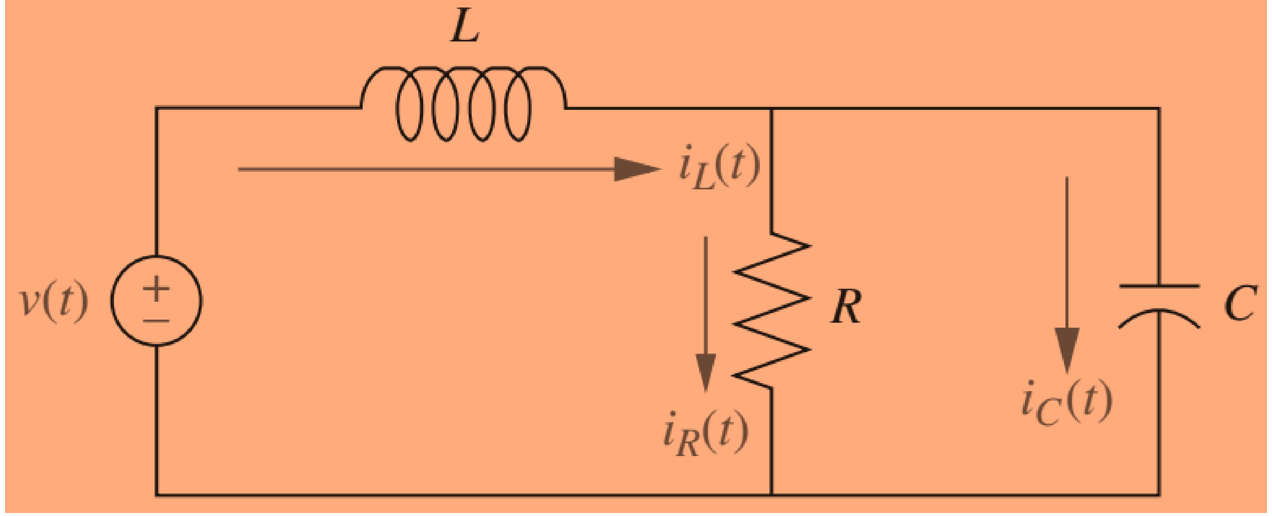
$\mathbf{x}$ = state vector
$\dot{\mathbf{x}}$ = derivative of the state vector with respect to time
$\mathbf{y}$ = output vector
$\mathbf{u}$ = input or control vector
$\mathbf{A}$ = system matrix
$\mathbf{B}$ = input matrix
$\mathbf{C}$ = output matrix
$\mathbf{D}$ = feedforward matrix

$$\frac{dx_1}{dt} = a_{11}x_1 + a_{12}x_2 + b_1 v(t)$$

$$\frac{dx_2}{dt} = a_{21}x_1 + a_{22}x_2 + b_2 v(t)$$

$$y = c_1 x_1 + c_2 x_2 + d_1 v(t)$$

Example

Given the electrical network , find a state-space representation if the output is the current through the resistor.

$$C\frac{dv_C}{dt} = i_C$$

$$L\frac{di_L}{dt} = v_L$$

$$i_C = -i_R + i_L$$

$$= -\frac{1}{R}v_C + i_L$$

$$v_L = -v_C + v(t)$$

$$C\frac{dv_C}{dt} = -\frac{1}{R}v_C + i_L \Rightarrow \frac{dv_C}{dt} = -\frac{1}{RC}v_C + \frac{1}{C}i_L$$

$$L\frac{di_L}{dt} = \quad -v_C + v(t) \Rightarrow \frac{di_L}{dt} = \quad -\frac{1}{L}v_C + \frac{1}{L}v(t)$$

continued...

$$\frac{dv_C}{dt} = -\frac{1}{RC}v_C + \frac{1}{C}i_L$$

$$\frac{di_L}{dt} = \quad -\frac{1}{L}v_C + \frac{1}{L}v(t)$$

$$i_R = \frac{1}{R}v_C$$

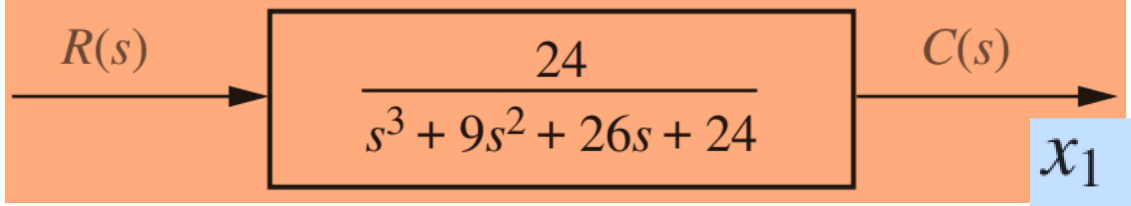
$$\begin{bmatrix} \dot{v}_C \\ \dot{i}_L \end{bmatrix} = \begin{bmatrix} -1/(RC) & 1/C \\ -1/L & 0 \end{bmatrix} \begin{bmatrix} v_C \\ i_L \end{bmatrix} + \begin{bmatrix} 0 \\ 1/L \end{bmatrix} v(t)$$

$$i_R = \begin{bmatrix} 1/R & 0 \end{bmatrix} \begin{bmatrix} v_C \\ i_L \end{bmatrix}$$

continued...

Find the state-space representation

$$\frac{C(s)}{R(s)} = \frac{24}{(s^3 + 9s^2 + 26s + 24)}$$



$$(s^3 + 9s^2 + 26s + 24)C(s) = 24R(s)$$

$$\dddot{c} + 9\ddot{c} + 26\dot{c} + 24c = 24r$$

Choosing the state variables

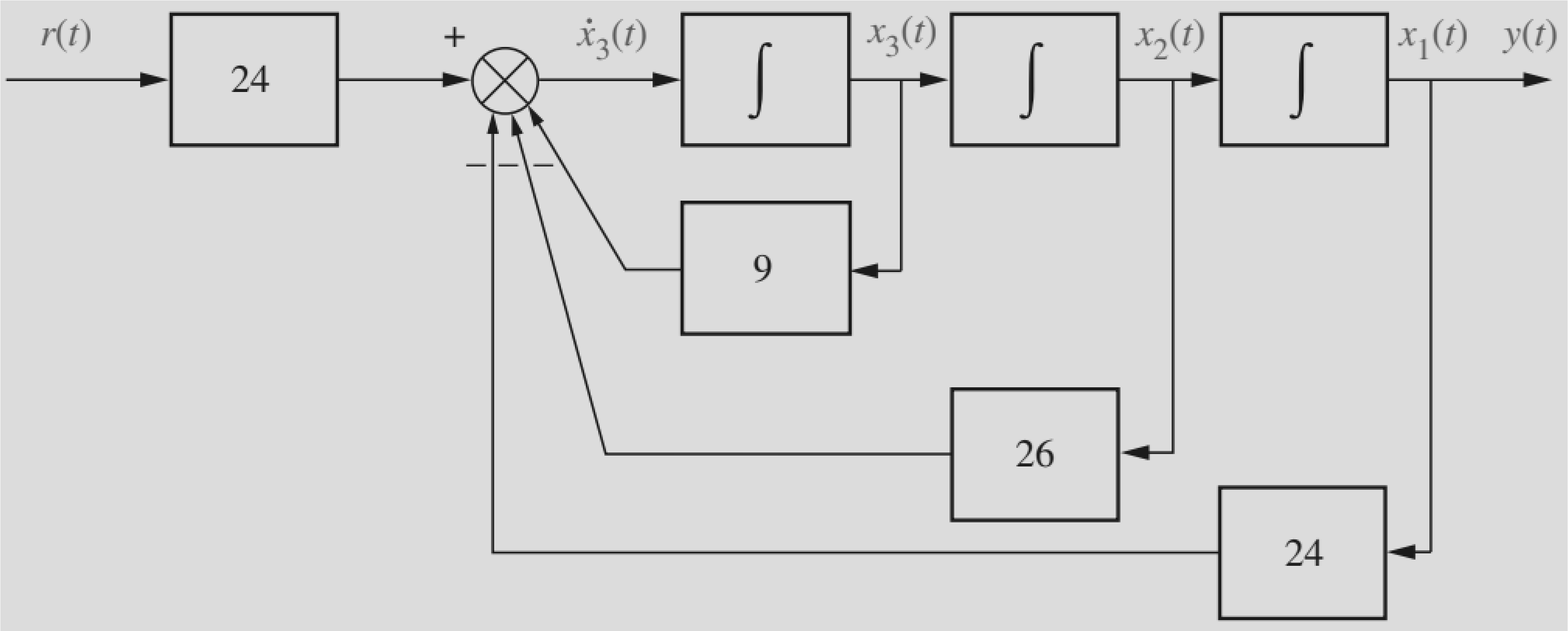
$$x_1 = c$$
$$x_2 = \dot{c}$$
$$x_3 = \ddot{c}$$

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3$$
$$\dot{x}_3 = -24x_1 - 26x_2 - 9x_3 + 24r$$
$$y = c = x_1$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 24 \end{bmatrix} r$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

continued…



$$y = c = x_1$$

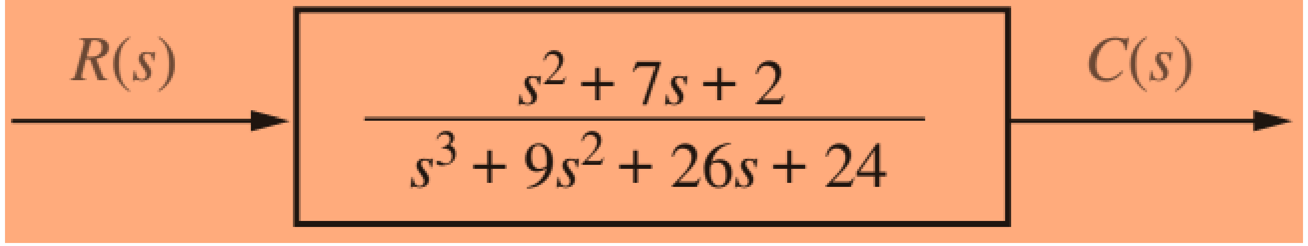
$$\dot{x}_1 = x_2$$

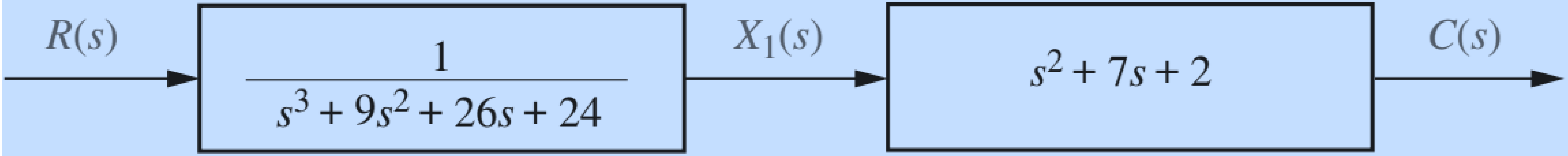
$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = -24x_1 - 26x_2 - 9x_3 + 24r$$

continued…

Find the state-space representation of the transfer function





$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r$$

$$C(s) = (s^2 + 7s + 2)X_1(s)$$

continued...

$$C(s) = (s^2 + 7s + 2)X_1(s)$$

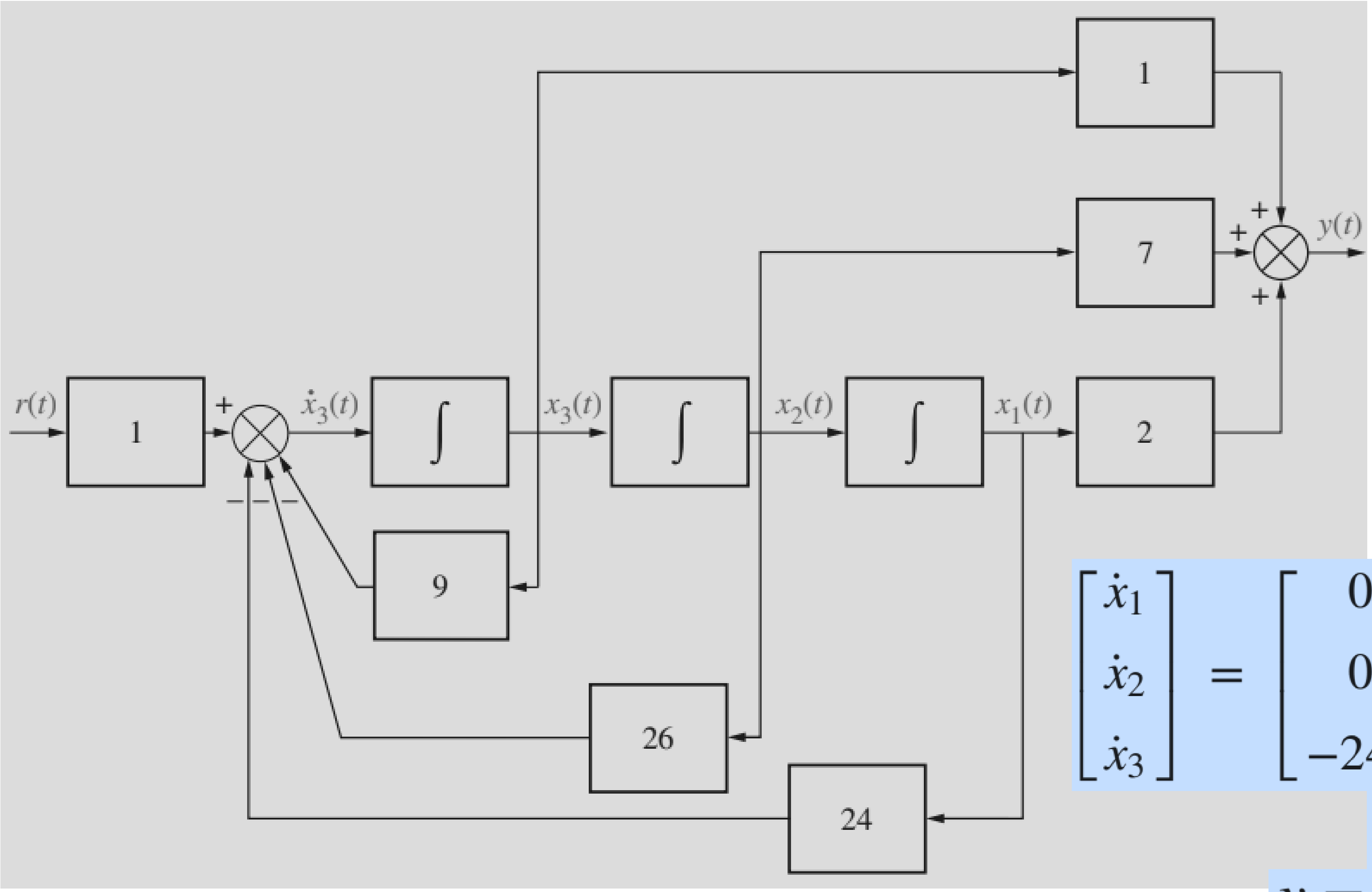
$$c = \ddot{x}_1 + 7\dot{x}_1 + 2x_1$$

$$\dot{x}_1 = x_2$$
$$\ddot{x}_1 = x_3$$

$$y = c(t) = x_3 + 7x_2 + 2x_1$$

$$y = \begin{bmatrix} 2 & 7 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

continued…



$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r$$

$$y = \begin{bmatrix} 2 & 7 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

continued...

Find the state equations and output equation of the transfer function $G(s) = \dfrac{2s+1}{s^2+7s+9}$.

$$\frac{X(s)}{R(s)} = \frac{1}{s^2+7s+9}$$

$$(s^2+7s+9)X(s) = R(s)$$

$$\ddot{x} + 7\dot{x} + 9x = r$$

Defining the state variables

$$x_1 = x$$

$$x_2 = \dot{x}$$

$$\dot{x}_2 = \ddot{x} = -7\dot{x} - 9x + r = -9x_1 - 7x_2 + r$$

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -9 & -7 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r(t)$$

$$c = 2\dot{x} + x = x_1 + 2x_2 \;\Rightarrow\; y = \begin{bmatrix} 1 & 2 \end{bmatrix}\mathbf{x}$$

State-Space Representation to Transfer Function

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$
$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

$$s\mathbf{X}(s) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s)$$
$$\mathbf{Y}(s) = \mathbf{C}\mathbf{X}(s) + \mathbf{D}\mathbf{U}(s)$$

$$(s\mathbf{I} - \mathbf{A})\mathbf{X}(s) = \mathbf{B}\mathbf{U}(s)$$

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s)$$

$$\mathbf{Y}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s) + \mathbf{D}\mathbf{U}(s)$$
$$= \left[\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}\right]\mathbf{U}(s)$$

$$T(s) = \frac{Y(s)}{U(s)} = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$

if $\mathbf{U}(s) = U(s)$ and $\mathbf{Y}(s) = Y(s)$

Example

find the transfer function,
$T(s) = Y(s)/U(s)$, where $U(s)$ is the input and $Y(s)$ is the output.

$$T(s) = \frac{Y(s)}{U(s)} = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix}u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\mathbf{x}$$

$$(s\mathbf{I} - \mathbf{A}) = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix} = \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1 & 2 & s+3 \end{bmatrix}$$

continued…

$$(s\mathbf{I}-\mathbf{A})^{-1} = \frac{\operatorname{adj}(s\mathbf{I}-\mathbf{A})}{\det(s\mathbf{I}-\mathbf{A})} = \frac{\begin{bmatrix} (s^2+3s+2) & s+3 & 1 \\ -1 & s(s+3) & s \\ -s & -(2s+1) & s^2 \end{bmatrix}}{s^3+3s^2+2s+1}$$

$$\mathbf{B} = \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

$$\mathbf{D} = 0$$

$$T(s) = \frac{Y(s)}{U(s)} = \mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$

$$T(s) = \frac{10(s^2+3s+2)}{s^3+3s^2+2s+1}$$

Time Response

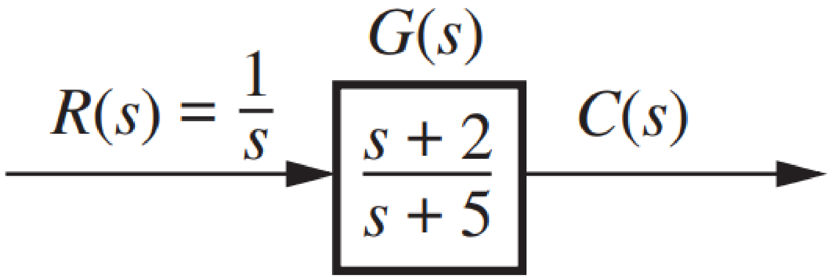
Output response = Forced response + Natural response

- Forced response = Steady-state response *or* Particular solution
- Natural response = Homogeneous solution

Poles: Values of the Laplace transform variable, s, that cause the transfer function to become **infinite**

Zeros: Values of the Laplace transform variable, s, that cause the transfer function to become **zero**

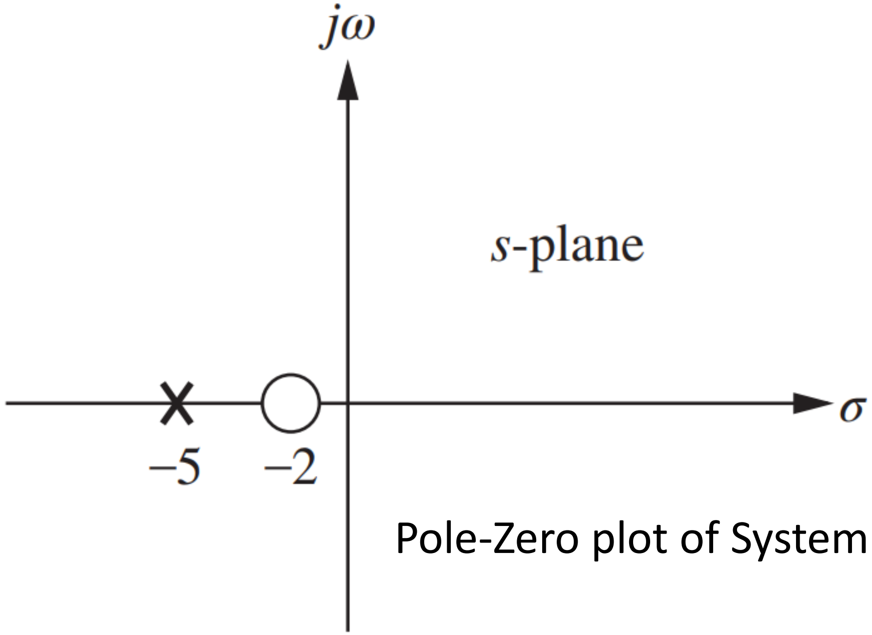
continued...



$$C(s) = \frac{(s+2)}{s(s+5)} = \frac{A}{s} + \frac{B}{s+5}$$

$$= \frac{2/5}{s} + \frac{3/5}{s+5}$$

$$c(t) = \frac{2}{5} + \frac{3}{5}e^{-5t}$$

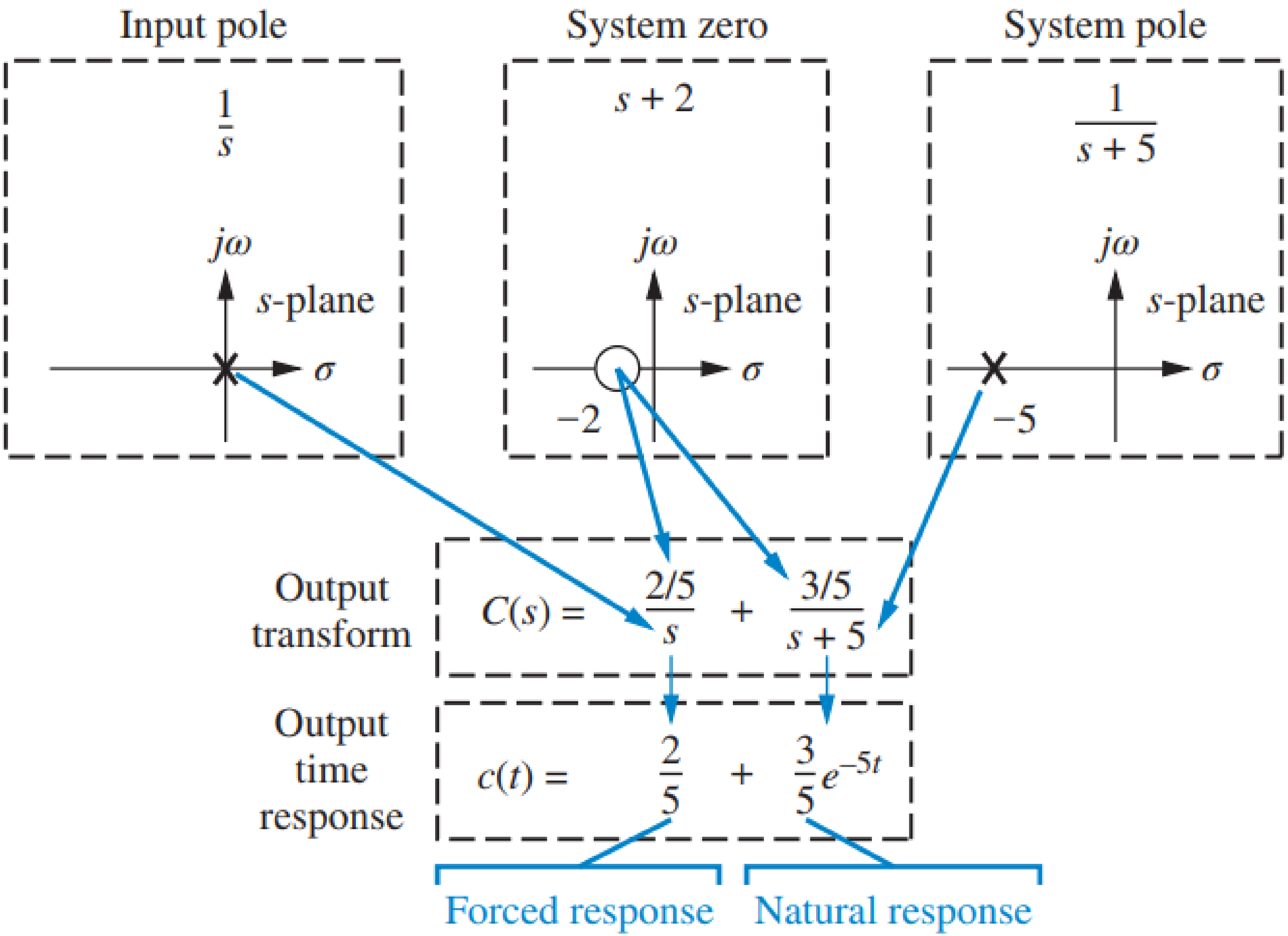


Pole-Zero plot of System

$$A = \left.\frac{(s+2)}{(s+5)}\right|_{s\to 0} = \frac{2}{5}$$

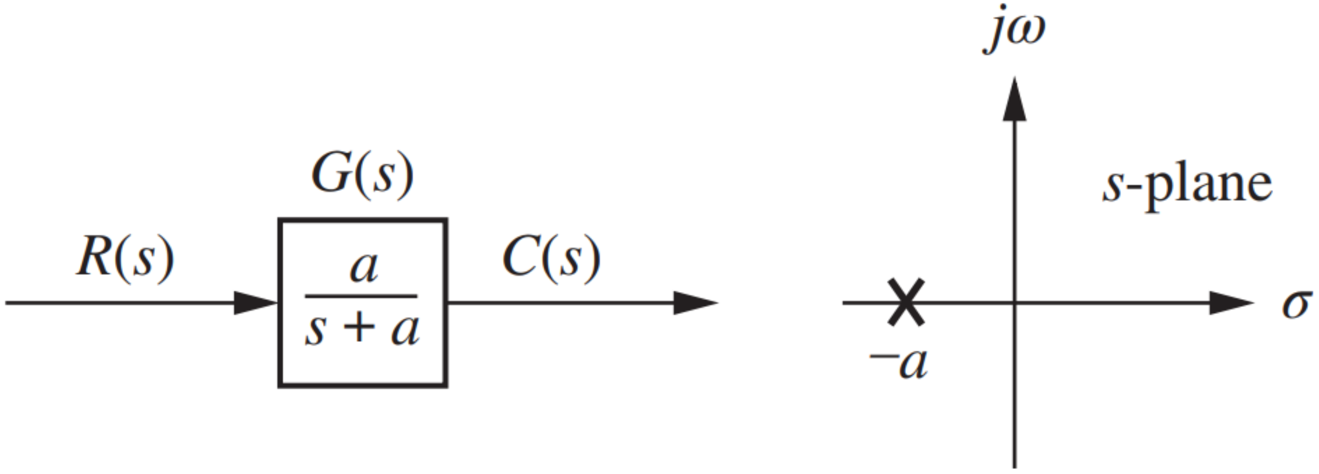
$$B = \left.\frac{(s+2)}{s}\right|_{s\to -5} = \frac{3}{5}$$

continued…



Input pole
$\frac{1}{s}$
$j\omega$
s-plane
σ
System zero
$s + 2$
$j\omega$
s-plane
σ
-2
System pole
$\frac{1}{s+5}$
$j\omega$
s-plane
σ
-5
Output transform
$C(s) = \frac{2/5}{s} + \frac{3/5}{s+5}$
Output time response
$c(t) = \frac{2}{5} + \frac{3}{5}e^{-5t}$
Forced response
Natural response

1st-order System

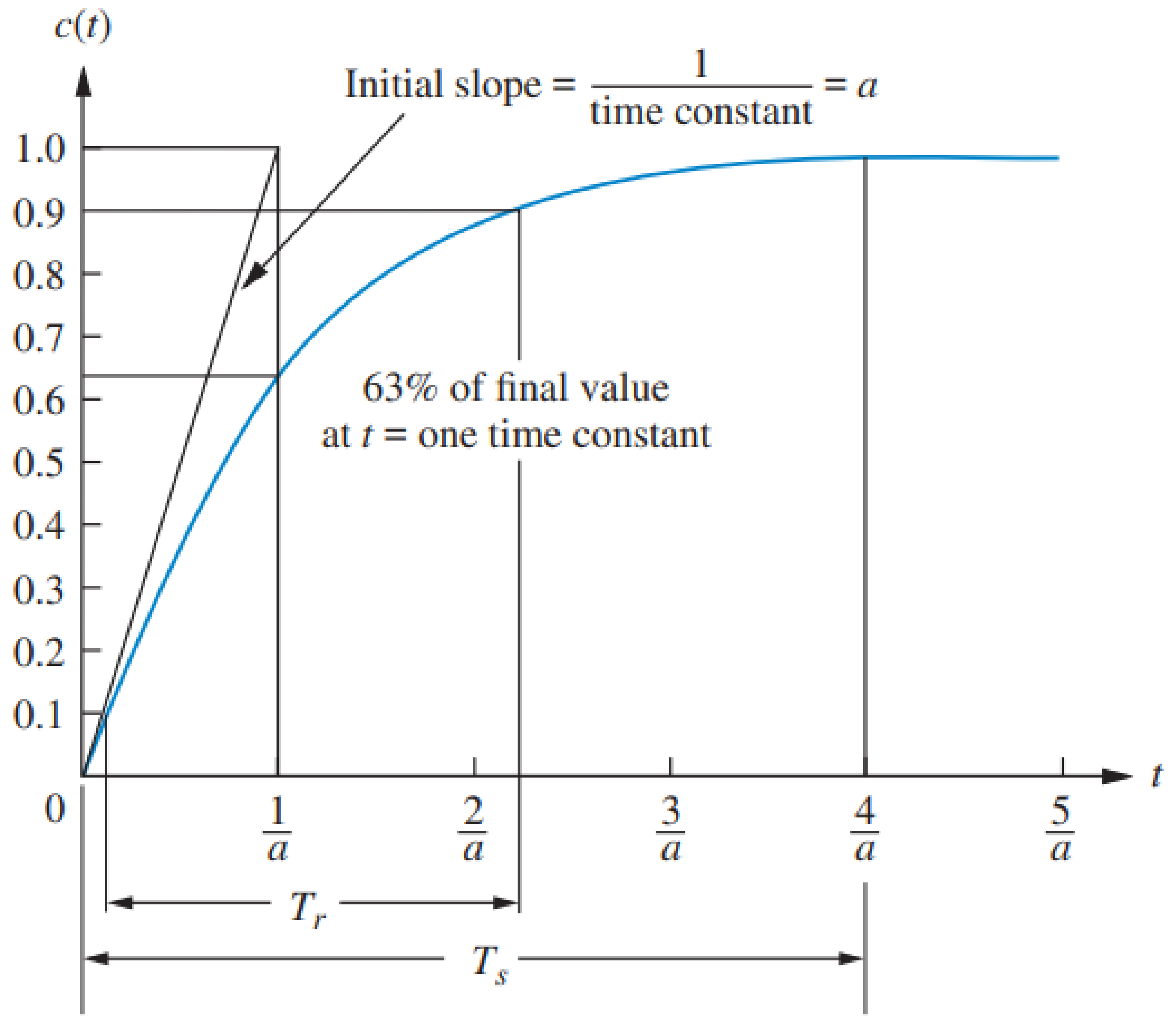


Laplace transform of the step response, $C(s) = R(s)G(s) = \dfrac{a}{s(s+a)}$

$$= \frac{1}{s} - \frac{1}{s+a}$$

$$c(t) = c_f(t) + c_n(t) = 1 - e^{-at}$$

continued...



$$\text{Rise Time, } T_r = \frac{2.2}{a}$$

[time for the waveform to go from 0.1 to 0.9 of its final value]

$$\text{Settling Time, } T_s = \frac{4}{a}$$

[time for the response to reach, and stay within, 2% of its final value]

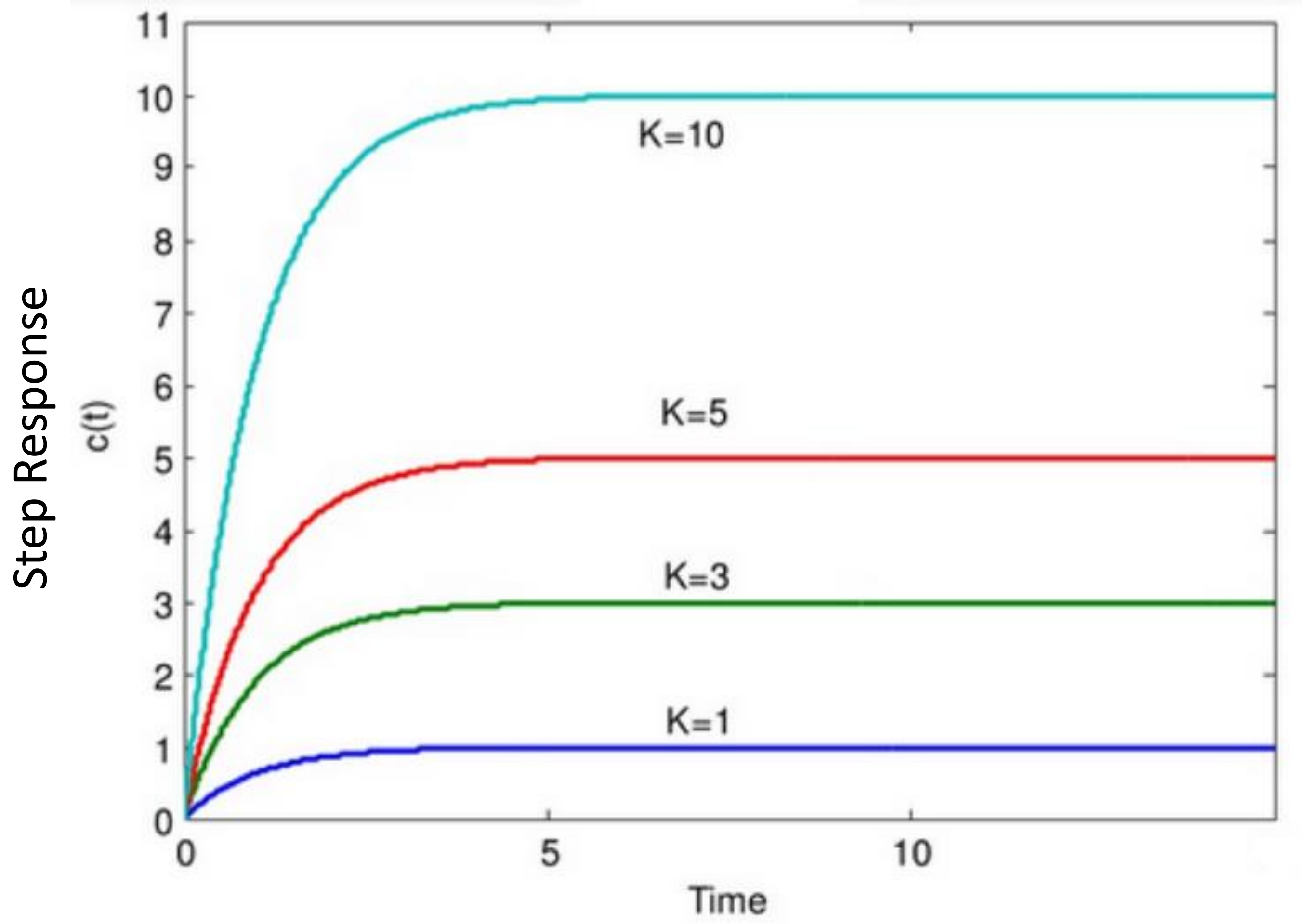
continued...

$$G(s) = \mathrm{b} / (s + a)$$

Laplace transform of the step response, $C(s) = \dfrac{\mathrm{b}}{s(s+a)} = \dfrac{\mathrm{b}/a}{s} - \dfrac{\mathrm{b}/a}{(s+a)}$

DC Gain, $K = \dfrac{b}{a}$

(also known as "Steady-state Value")



1st-order System with Zero

$$G(s) = \frac{\frac{20}{3}\left(s + \frac{1}{2}\right)}{s + \frac{1}{3}}$$

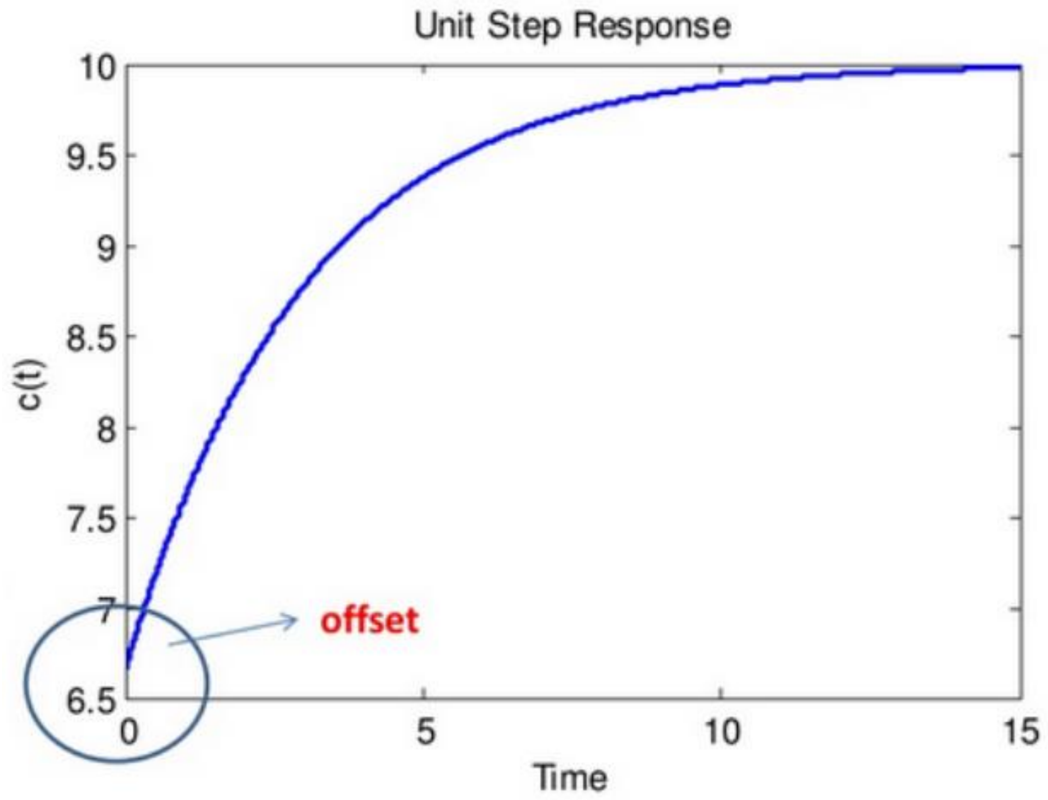
If input is step, $R(s) = {}^{1}/_{s}$

$$\therefore C(s) = R(s) \cdot G(s) = \frac{\frac{20}{3}\left(s + \frac{1}{2}\right)}{s\left(s + \frac{1}{3}\right)}$$

$$\Longrightarrow C(s) = \frac{10}{s} - \frac{{}^{10}/_{3}}{s + \frac{1}{3}}$$

$$\therefore c(t) = 10 - \frac{10}{3} e^{-t/3}$$

Case 1: Pole is nearer to imaginary axis



continued…

$$G(s) = \frac{\frac{40}{3}\left(s + \frac{1}{2}\right)}{s + \frac{2}{3}}$$

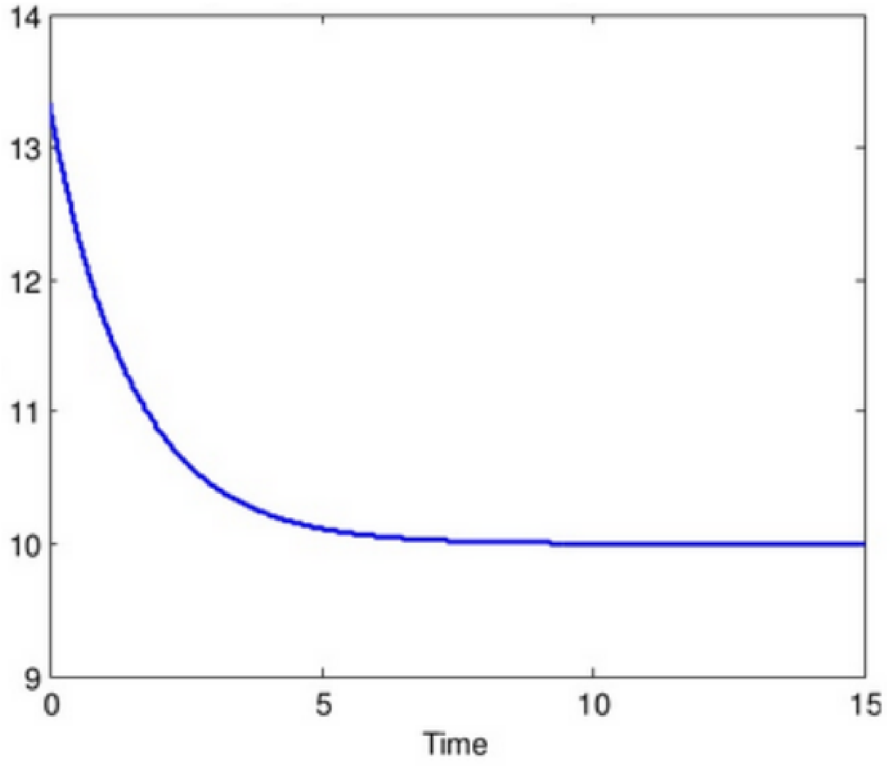
Case 2: Zero is nearer to imaginary axis

If input is step, $R(s) = {}^{1}/_{s}$

$$\therefore C(s) = R(s) \cdot G(s) = \frac{\frac{40}{3}\left(s + \frac{1}{2}\right)}{s\left(s + \frac{2}{3}\right)}$$

$$\Longrightarrow C(s) = \frac{10}{s} + \frac{{}^{10}/_{3}}{s + \frac{2}{3}}$$

$$\therefore c(t) = 10 + \frac{10}{3} e^{-2t/3}$$



2^{nd}-order System

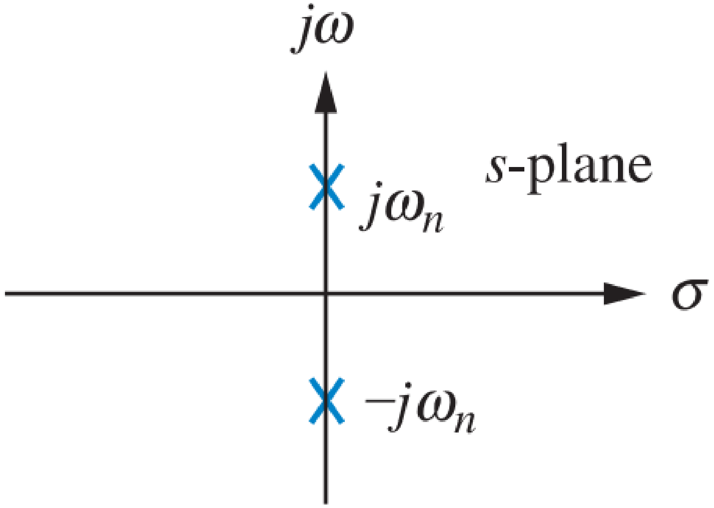
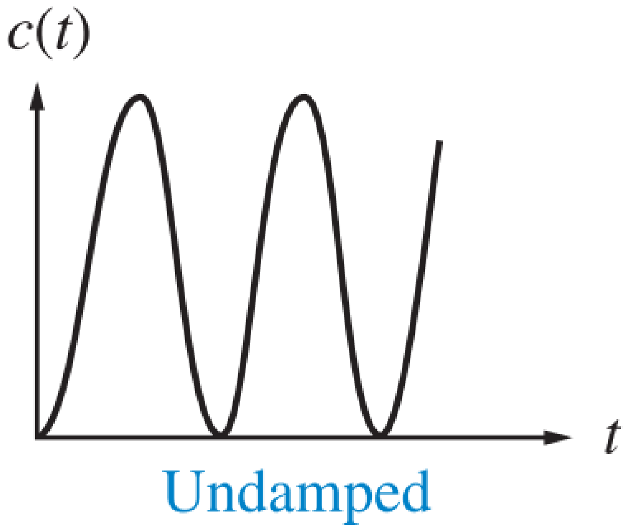
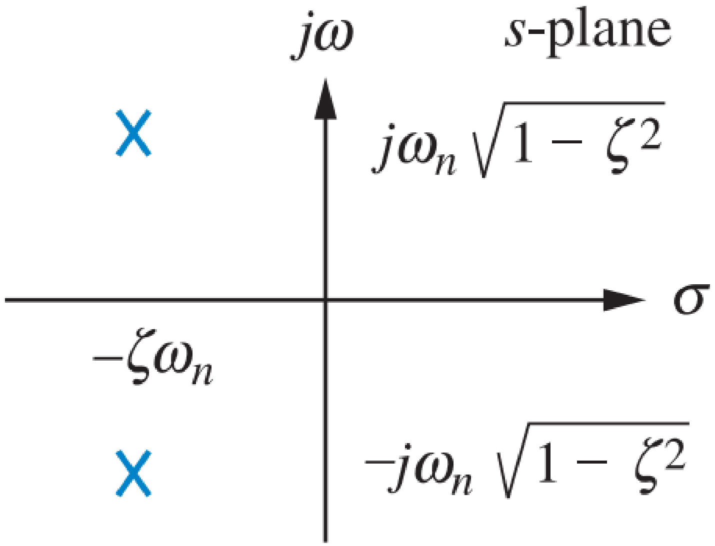
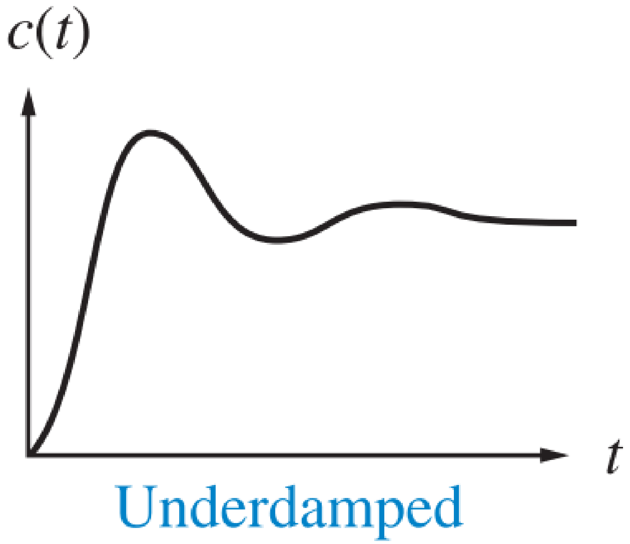
$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Natural Frequency, ω_n = frequency of oscillation without damping

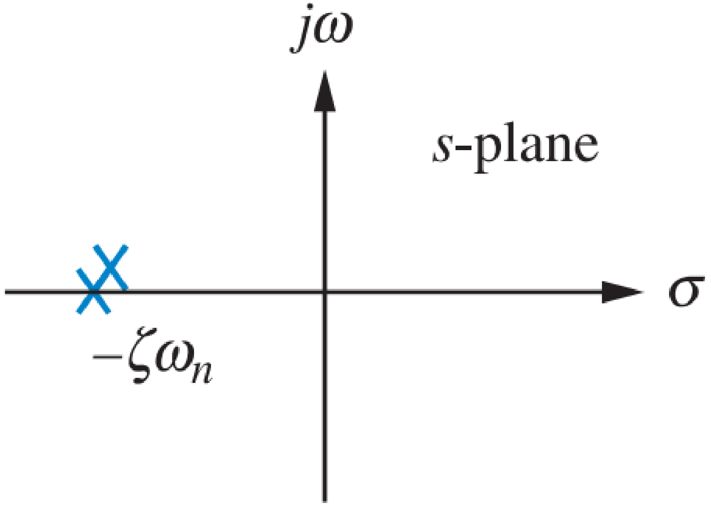
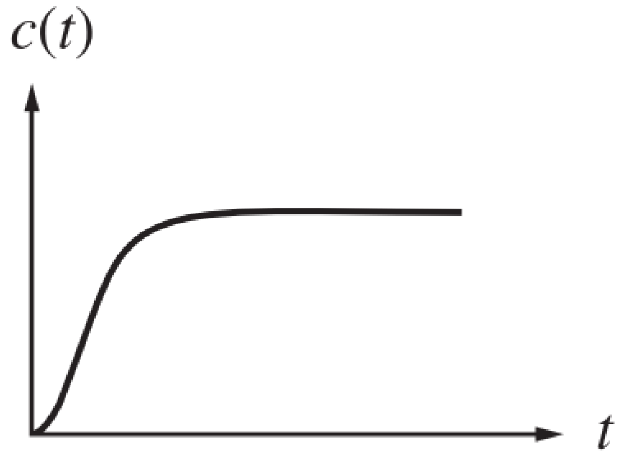
Damping Ratio, ζ = exponential decay frequency / natural frequency

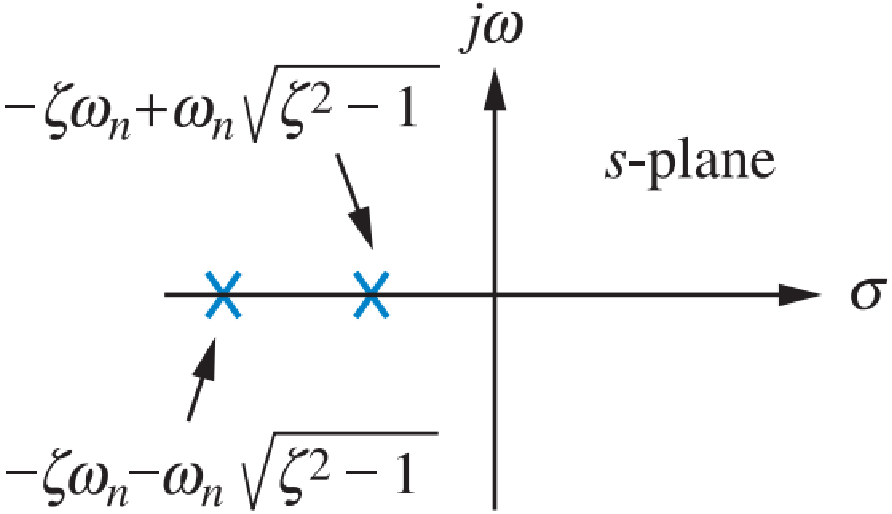
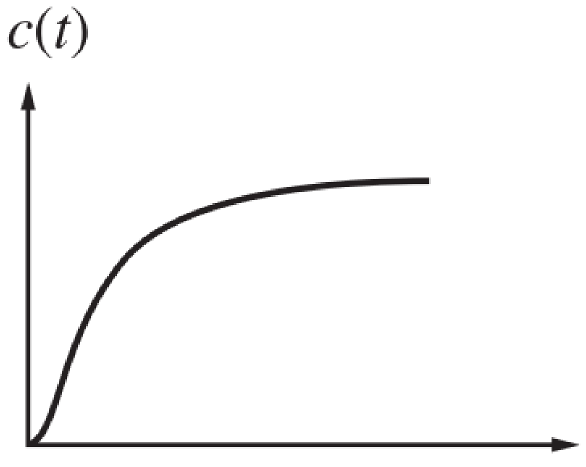
Pole Location: $s_{1,\,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$

continued…

ζ	Pole Location	Step Response
$\zeta = 0$		
$0 < \zeta < 1$		

continued…

<table>
<tr><th>ζ</th><th>Pole Location</th><th>Step Response</th></tr>
<tr><td>$\zeta = 1$</td><td>
</td><td>

Critically damped</td></tr>
<tr><td>$\zeta > 1$</td><td>
</td><td>

Overdamped</td></tr>
</table>

Example

Find the values of ζ and ω_n, characterize the nature of the response.

a. $G(s) = \dfrac{400}{s^2 + 12s + 400}$

b. $G(s) = \dfrac{900}{s^2 + 90s + 900}$

c. $G(s) = \dfrac{225}{s^2 + 30s + 225}$

d. $G(s) = \dfrac{625}{s^2 + 625}$

a. $\zeta = 0.3,\ \omega_n = 20$; underdamped

b. $\zeta = 1.5,\ \omega_n = 30$; overdamped

c. $\zeta = 1,\ \omega_n = 15$; critically damped

d. $\zeta = 0,\ \omega_n = 25$; undamped

Underdamped 2nd-order System

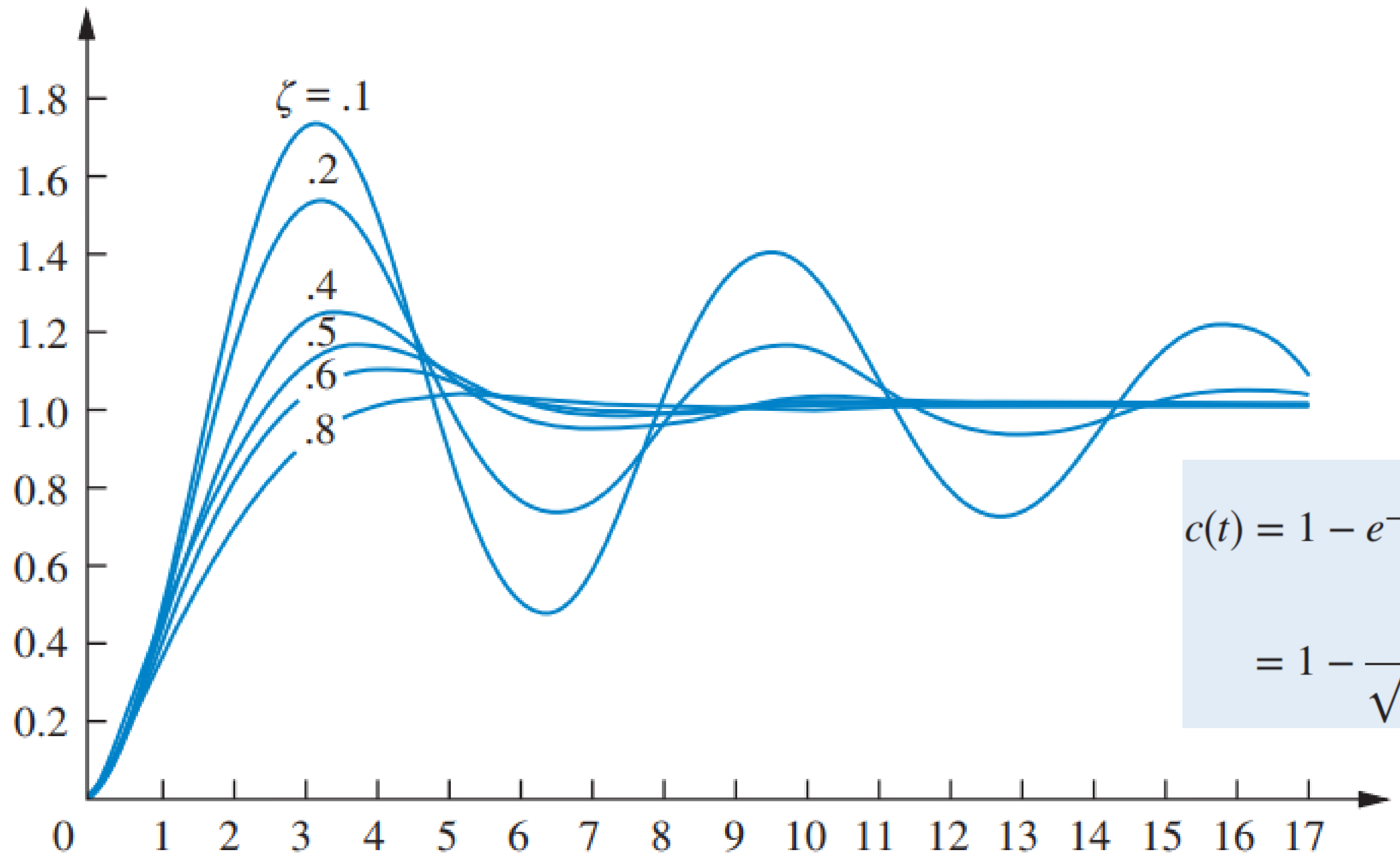
Laplace transform of the step response, $C(s) = \dfrac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}$

$$= \frac{K_1}{s} + \frac{K_2 s + K_3}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$c(t) = 1 - e^{-\zeta\omega_n t}\left(\cos \omega_n\sqrt{1-\zeta^2}\,t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin \omega_n\sqrt{1-\zeta^2}\,t\right)$$

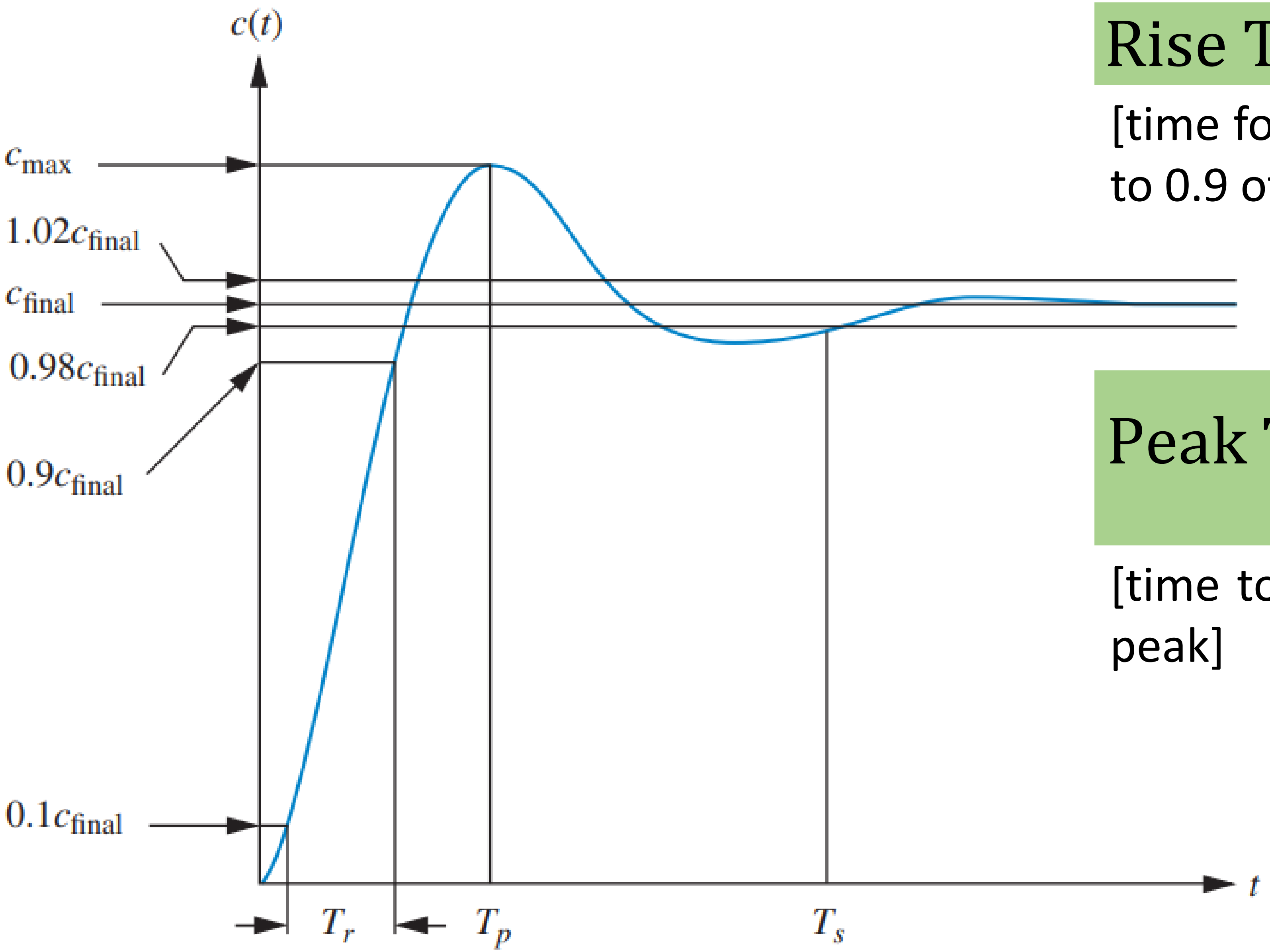
$$= 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\cos(\omega_n\sqrt{1-\zeta^2}\,t - \phi)$$

where $\phi = \tan^{-1}(\zeta/\sqrt{1-\zeta^2})$



2nd-order (underdamped) step responses for damping ratio values

continued...



Rise Time, T_r

[time for the waveform to go from 0.1 to 0.9 of its final value]

Peak Time, $T_p = \dfrac{\pi}{\omega_n\sqrt{1-\zeta^2}}$

[time to reach the first, or maximum, peak]

continued…

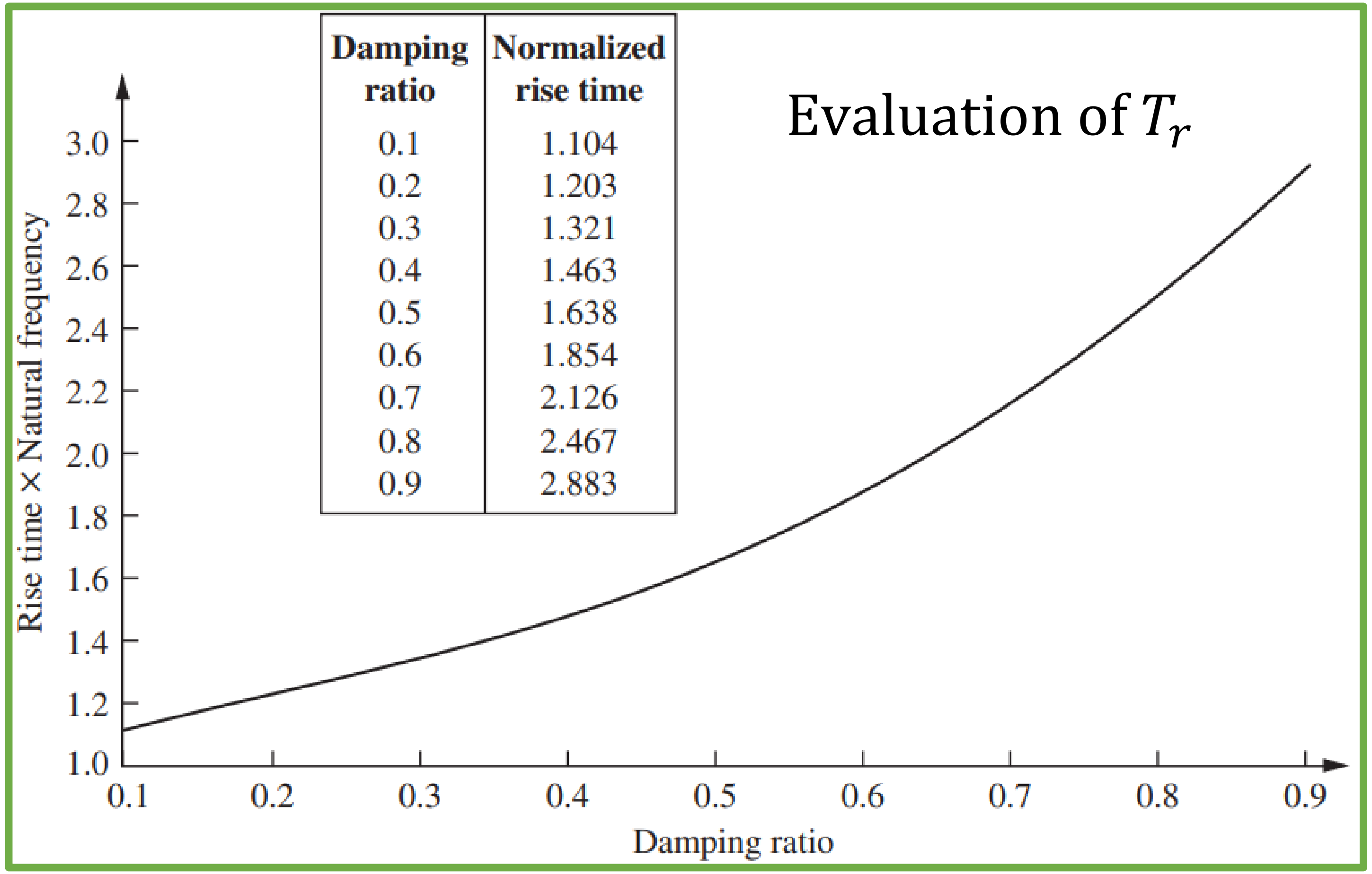
$$\text{Percent Overshoot, } \%OS = e^{-\left(\zeta\pi/\sqrt{1-\zeta^2}\right)} \times 100$$

$$= \frac{c_{\max} - c_{\text{final}}}{c_{\text{final}}} \times 100$$

$$\text{Settling Time, } T_s = \frac{4}{\zeta\omega_n}$$

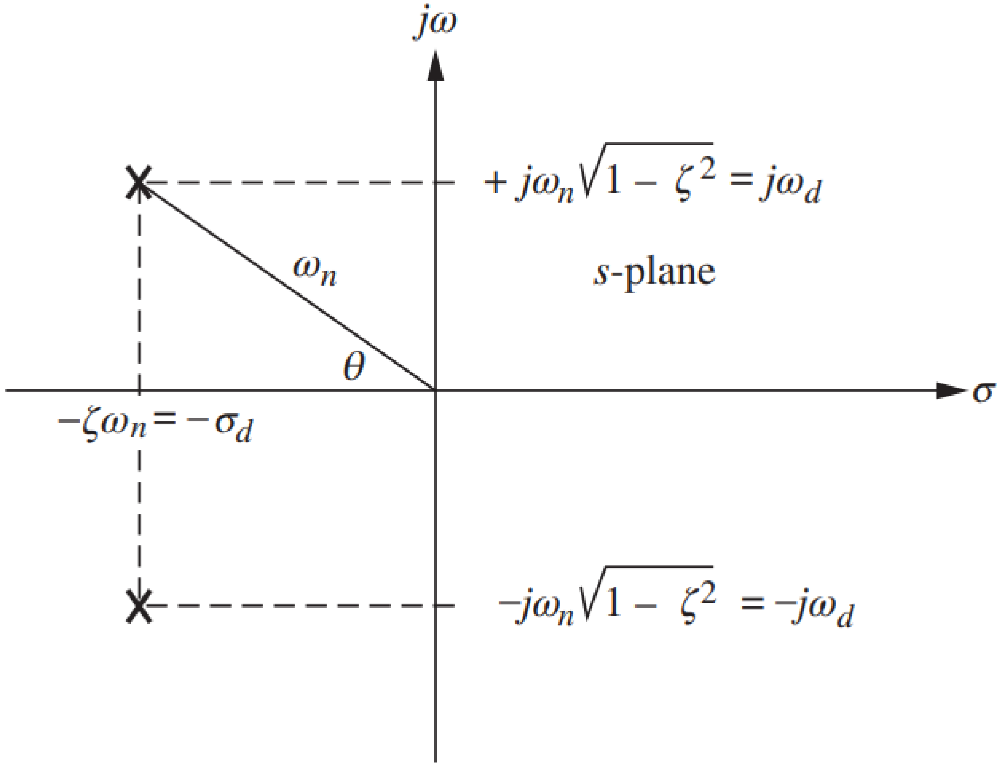
[time for transient's damped oscillations to reach and stay within ±2% of the steady-state value]

Evaluation of T_r



Damping ratio	Normalized rise time
0.1	1.104
0.2	1.203
0.3	1.321
0.4	1.463
0.5	1.638
0.6	1.854
0.7	2.126
0.8	2.467
0.9	2.883

continued…



$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{\omega_d}$$

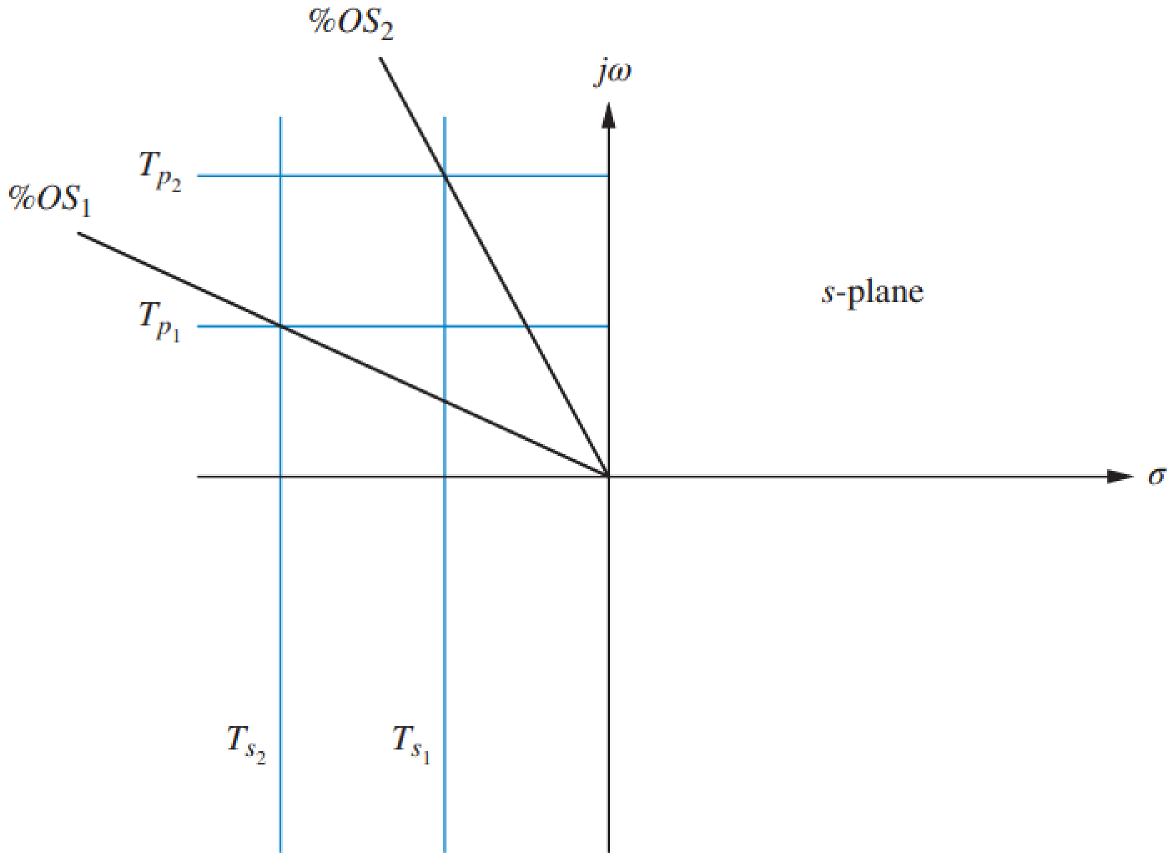
$$T_s = \frac{4}{\zeta\omega_n} = \frac{4}{\sigma_d}$$

ω_d = damped frequency of oscillation

σ_d = exponential damping frequency

$\zeta = \cos\theta$

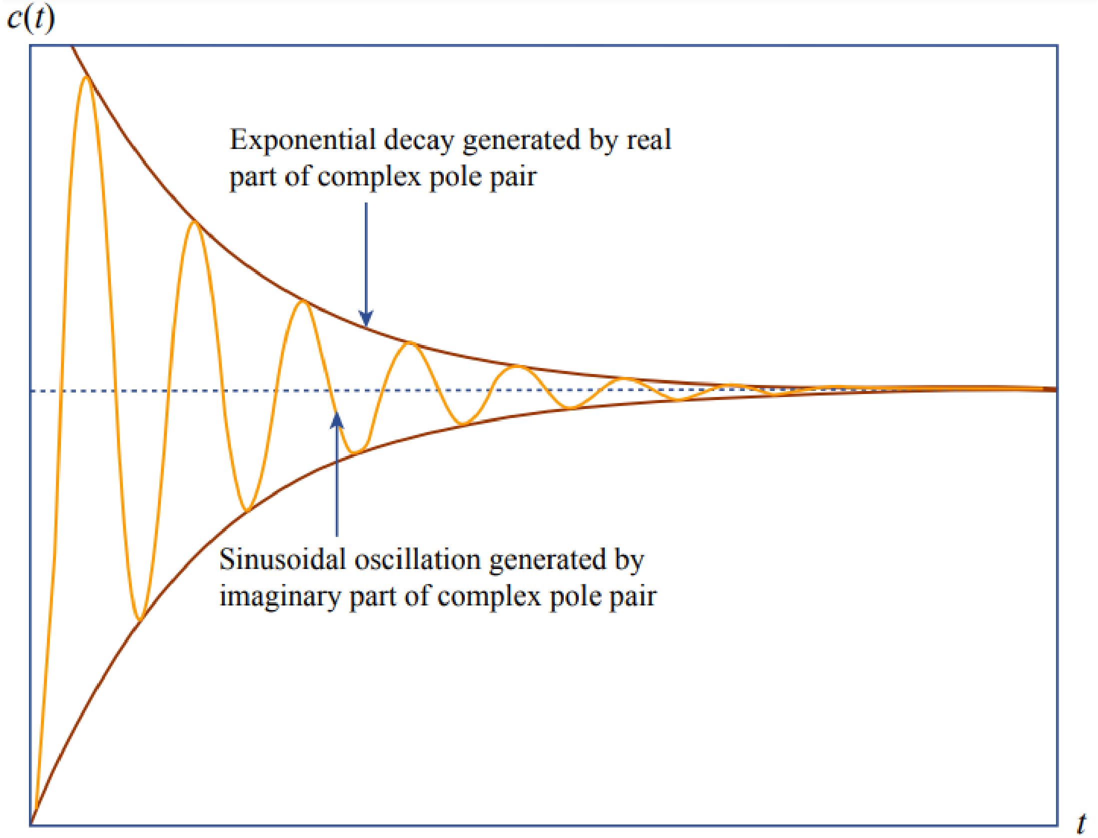
continued...



$$T_{s_2} < T_{s_1}$$

$$T_{p2} < T_{p1}$$

$$\%OS_1 < \%OS_2$$

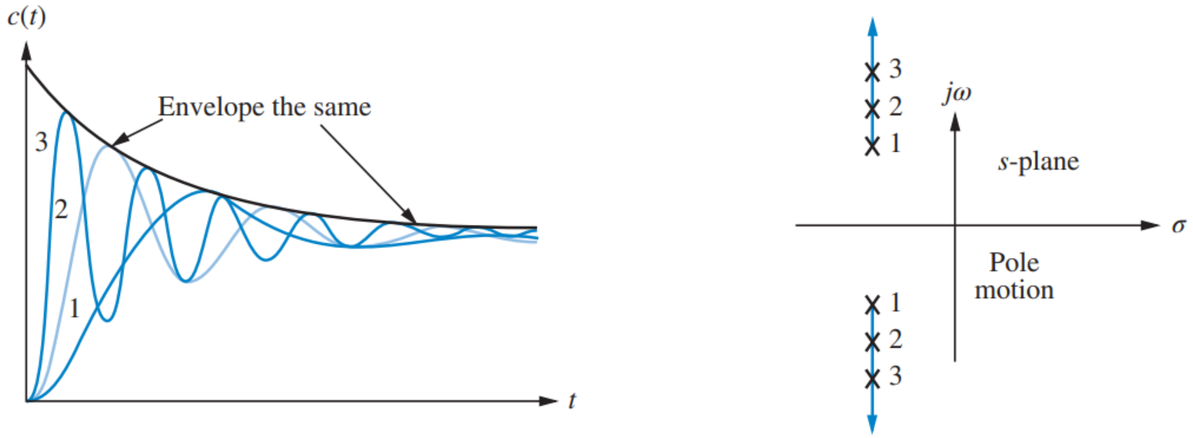


ω_d = damped frequency of oscillation (imaginary part)

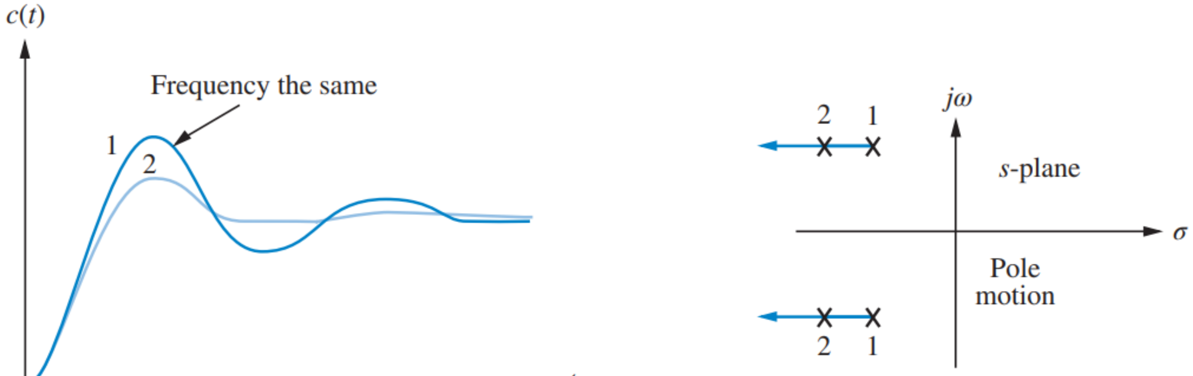
σ_d = exponential damping frequency (real part)

continued…

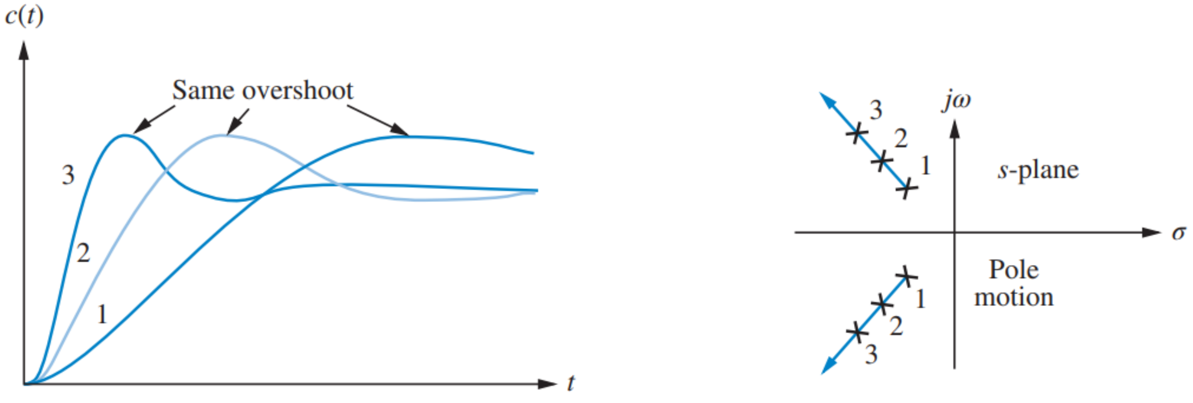
<u>$1 \longrightarrow 2 \longrightarrow 3$</u>



Settling time is same.
Peak time decreases.
Overshoot increases.
Rise time decreases.



Settling time decreases.
Peak time is same.
Overshoot decreases.
Rise time increases.



Settling time decreases.
Peak time decreases.
Overshoot is same.
Rise time decreases.

Example

find T_p, $\%OS$, T_s, and T_r

$$G(s) = \frac{100}{s^2 + 15s + 100}$$

$\omega_n = 10$

$\zeta = 0.75$

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = 0.48$$

$$\%OS = e^{-\left(\zeta\pi/\sqrt{1-\zeta^2}\right)} \times 100 = 2.83$$

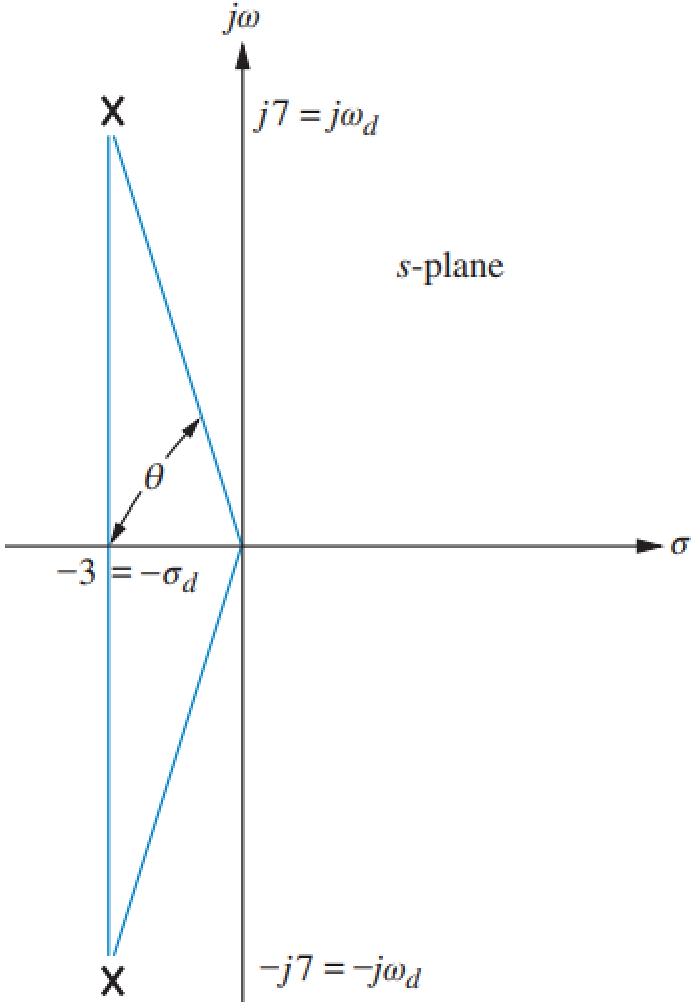
$$T_s = \frac{4}{\zeta\omega_n} = 0.53$$

Normalized $T_r = 2.3$ (from Table)

$$\text{Actual } T_r = \frac{\text{Normalized } T_r}{\omega_n} = 0.23$$

continued...

Find damping ratio, natural frequency, peak time, overshoot and settling time for an underdamped 2nd-order system, if the poles are at $-3 \pm j7$.



$$\zeta = \cos\theta = \cos\left(\tan^{-1}\frac{7}{3}\right) = 0.394$$

$$\omega_n = \sqrt{7^2 + 3^2} = 7.616$$

$$T_p = \frac{\pi}{\omega_d} = \frac{\pi}{7} = 0.449$$

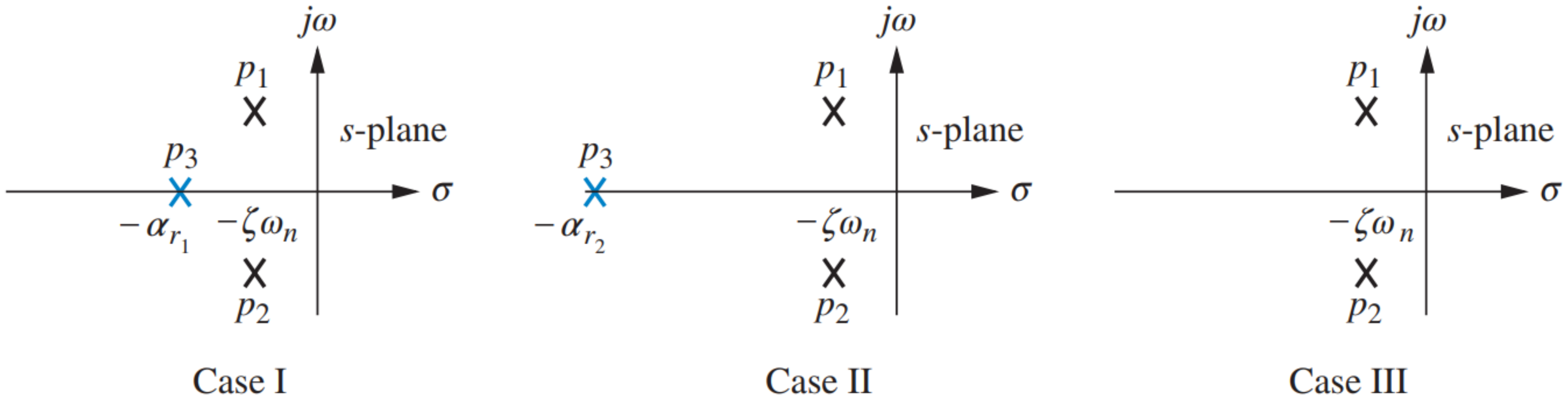
$$\text{Overshoot} = e^{-\left(\zeta\pi/\sqrt{1-\zeta^2}\right)} \times 100 = 26\%$$

$$T_s = \frac{4}{\sigma_d} = \frac{4}{3} = 1.333$$

2^{nd}-order Approximation from 3^{rd}-order System

Complex pole at: $-\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$

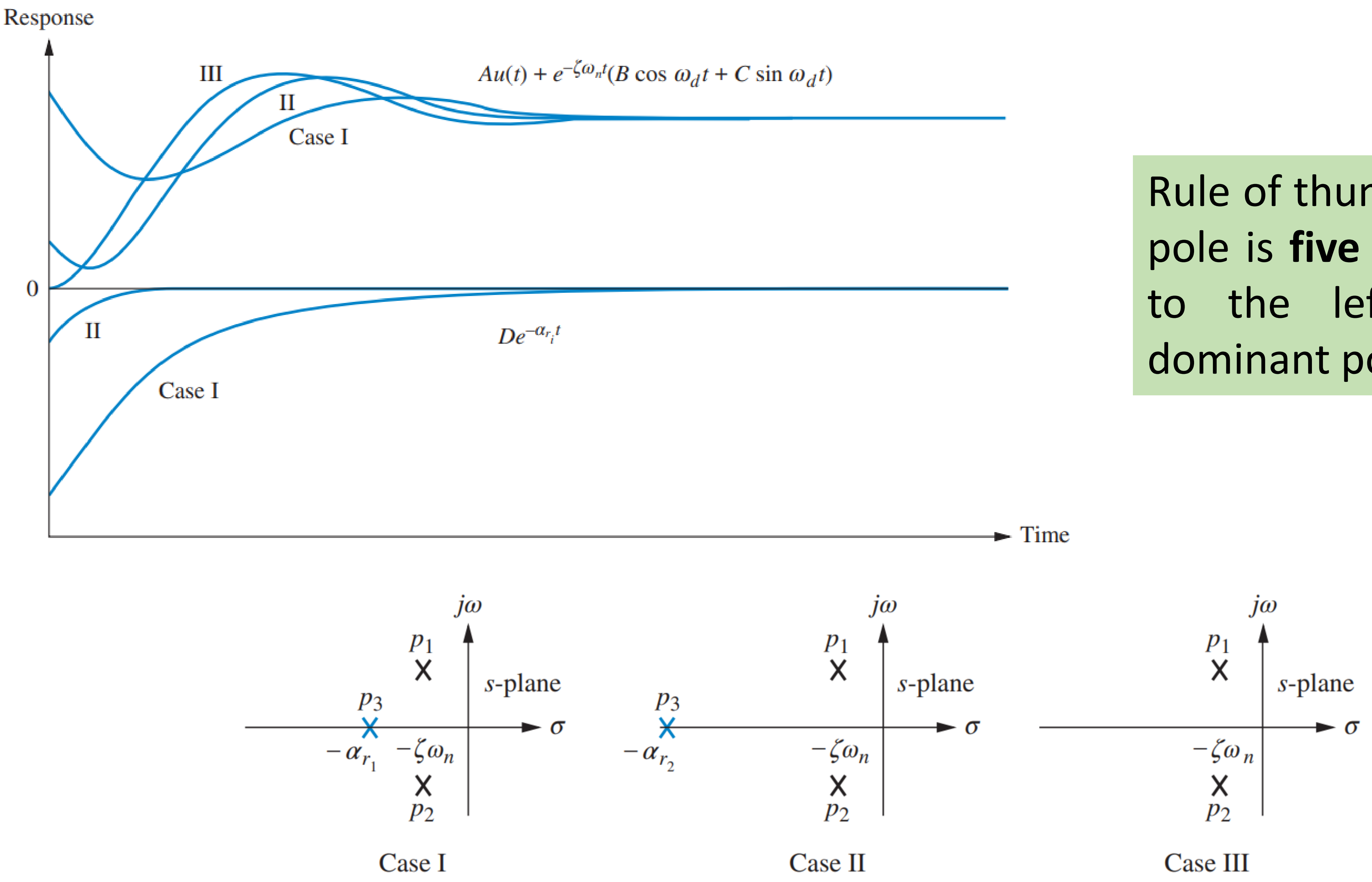
Real pole at: $-\alpha_r$



Laplace transform of the step response, $C(s) = \frac{A}{s} + \frac{B(s+\zeta\omega_n) + C\omega_d}{(s+\zeta\omega_n)^2 + \omega_d^2} + \frac{D}{s+\alpha_r}$

$$c(t) = Au(t) + e^{-\zeta\omega_n t}(B\cos\omega_d t + C\sin\omega_d t) + De^{-\alpha_r t}$$

continued…



Rule of thumb: if the real pole is **five** times farther to the left than the dominant poles

Example

Determine the validity of a 2nd-order approximation for each of these two transfer functions:

a. $G(s) = \dfrac{700}{(s+15)(s^2+4s+100)}$

b. $G(s) = \dfrac{360}{(s+4)(s^2+2s+90)}$

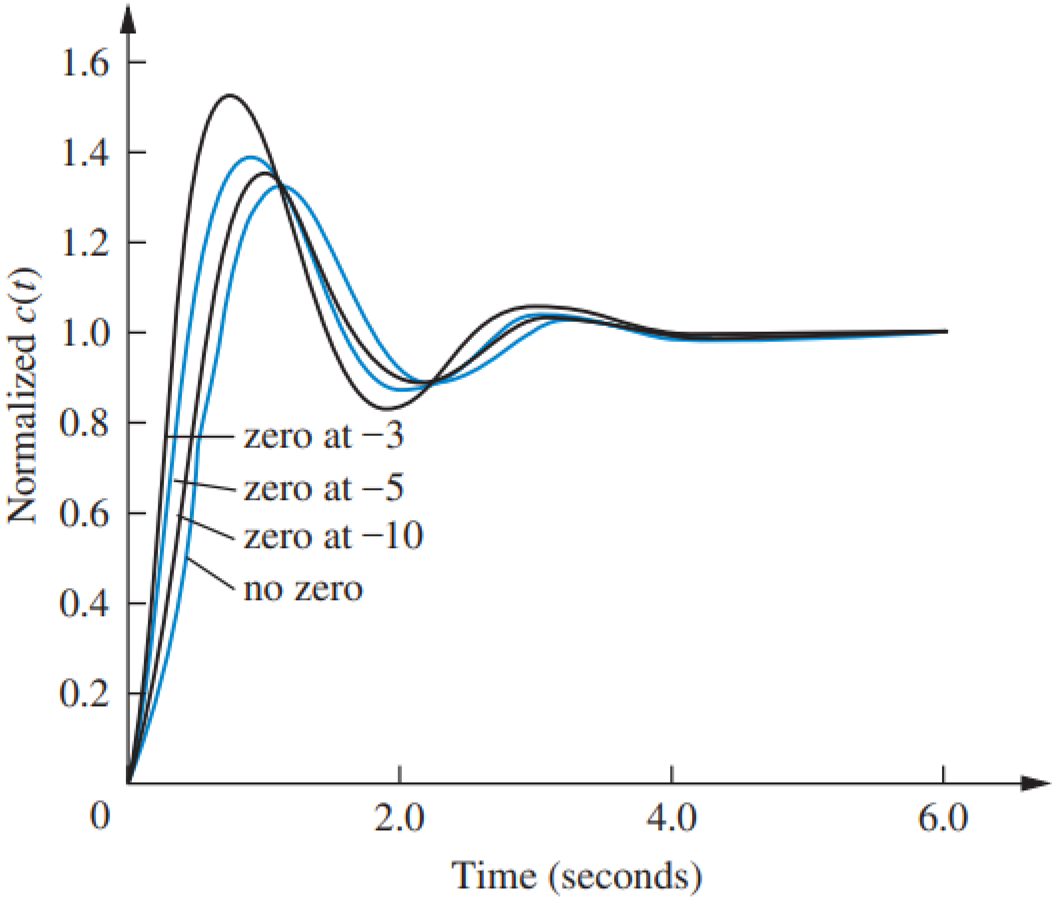
a. Complex poles at $-2 \pm j9.8$
Real pole at -15
More than five-times further.
So, approximation will be valid.

b. Complex poles at $-1 \pm j9.4$
Real pole at -4
Not more than five-times further.
So, approximation will be invalid.

Effect of Adding a Zero

$$(s+a)C(s) = sC(s) + aC(s)$$

- If ‘a’ is very large, the Laplace transform of the response is approximately $aC(s)$, or a scaled version of the original response.
- If ‘a’ becomes smaller, the derivative term contributes more to the response and has a greater effect.



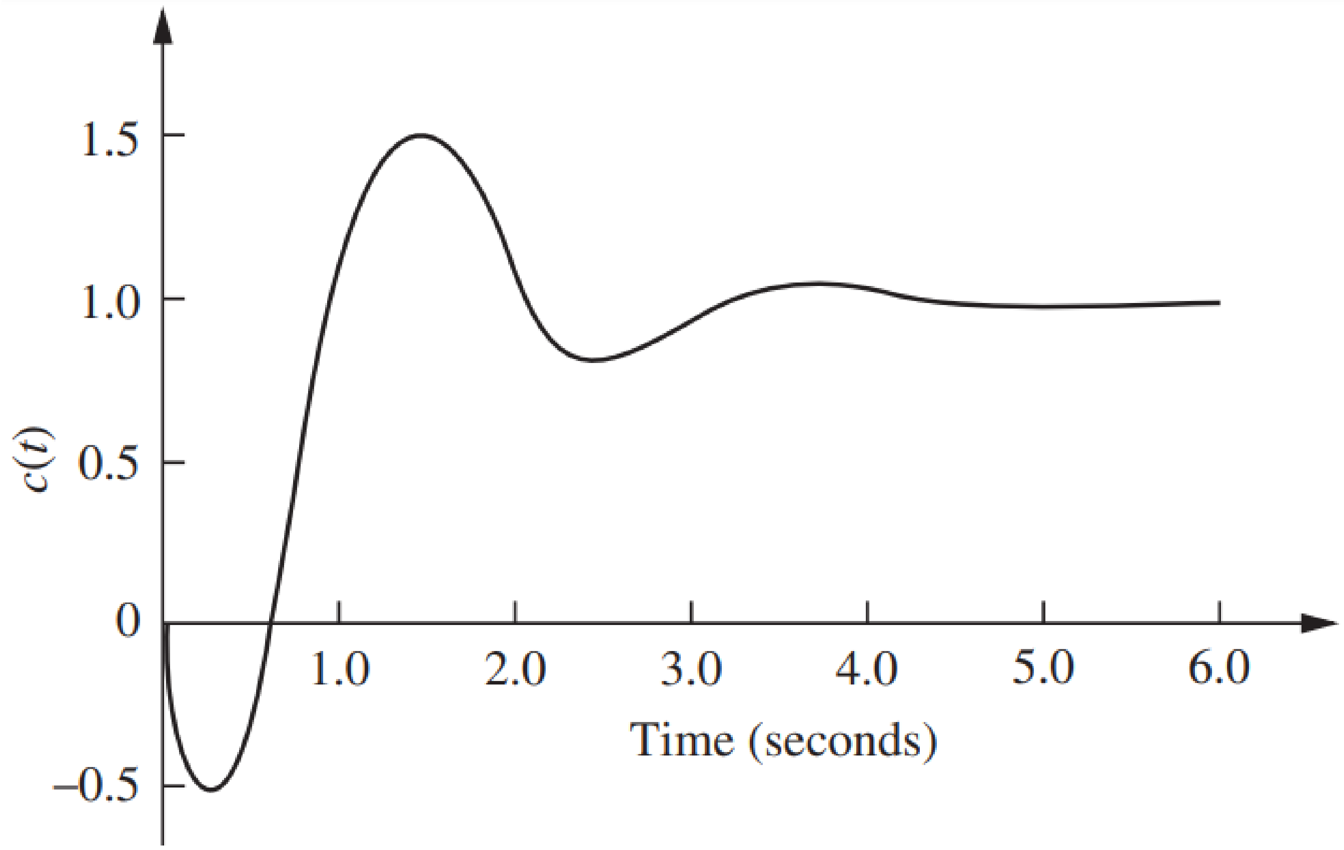
Effect of adding a zero to a 2-pole system

continued…

What if ‘a’ is negative?

The zero is in the right half-plane.

Nonminimum-phase system: response begins to turn toward the opposite direction of the final value.



Pole-Zero Cancellation

$$T(s) = \frac{K(s+z)}{(s+p_3)(s^2+as+b)}$$

$$C_1(s) = \frac{26.25(s+4)}{s(s+3.5)(s+5)(s+6)}$$

$$C_1(s) = \frac{1}{s} - \frac{3.5}{s+5} + \frac{3.5}{s+6} - \frac{1}{s+3.5}$$

$$C_2(s) = \frac{26.25(s+4)}{s(s+4.01)(s+5)(s+6)}$$

$$C_2(s) = \frac{0.87}{s} - \frac{5.3}{s+5} + \frac{4.4}{s+6} + \frac{0.033}{s+4.01}$$

$$C_2(s) \approx \frac{0.87}{s} - \frac{5.3}{s+5} + \frac{4.4}{s+6}$$

$$c_2(t) \approx 0.87 - 5.3e^{-5t} + 4.4e^{-6t}$$

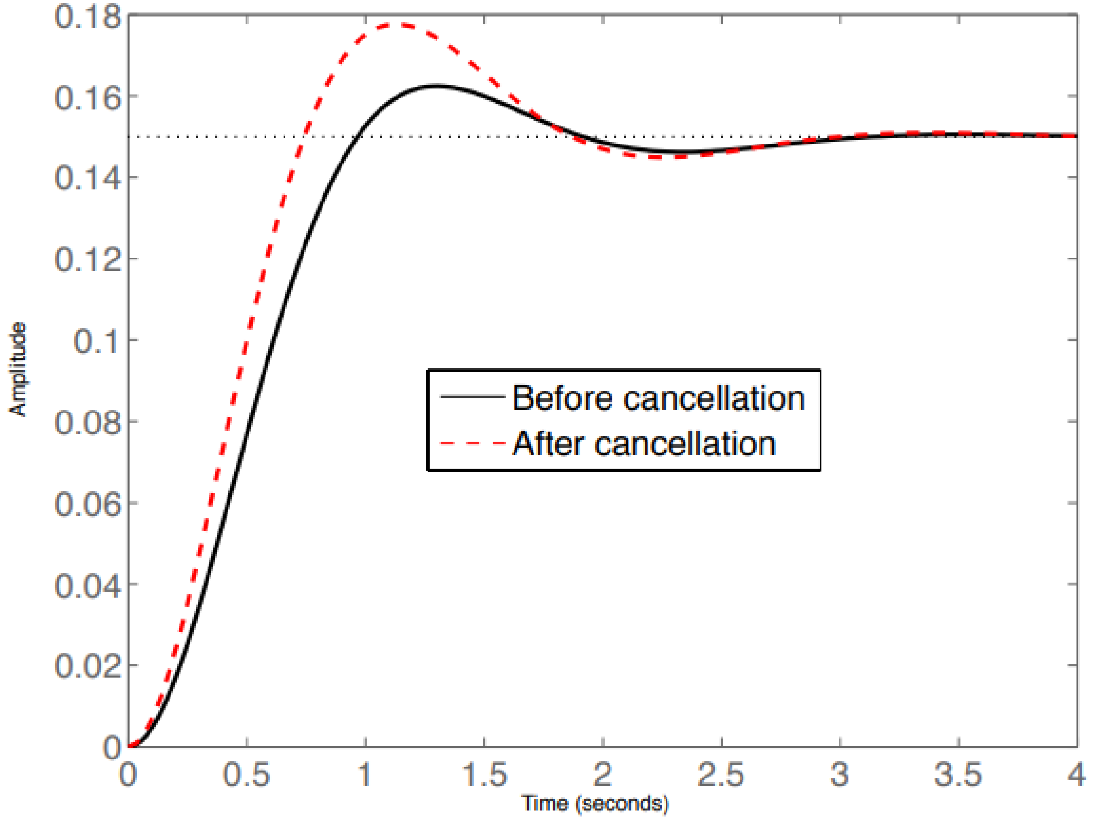
continued…

(a) $$G(s) = \frac{s+3}{(s+2)(s^2+3s+10)}$$

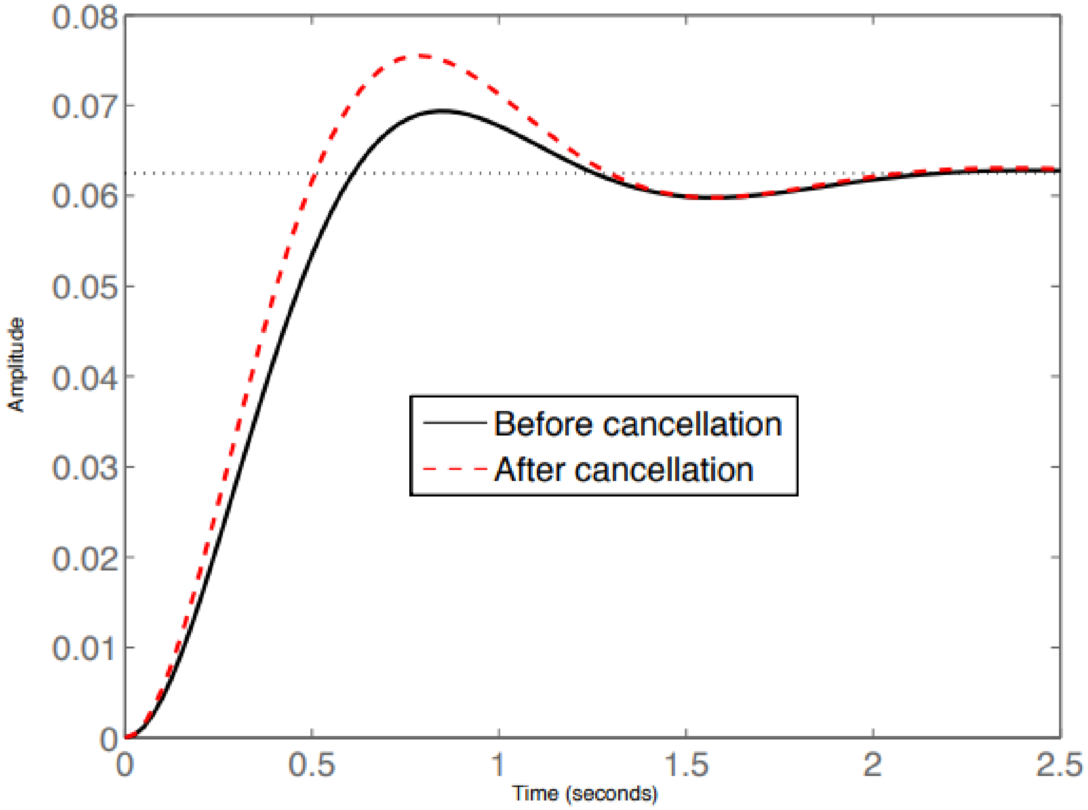
(b) $$G(s) = \frac{s+2.5}{(s+2)(s^2+4s+20)}$$

$$G_c(s) = \frac{1.5}{s^2+3s+10}$$

$$G_c(s) = \frac{1.25}{s^2+4s+20}$$



(a) Difference in response; probably shouldn't cancel.



(b) A little less difference, but still some.

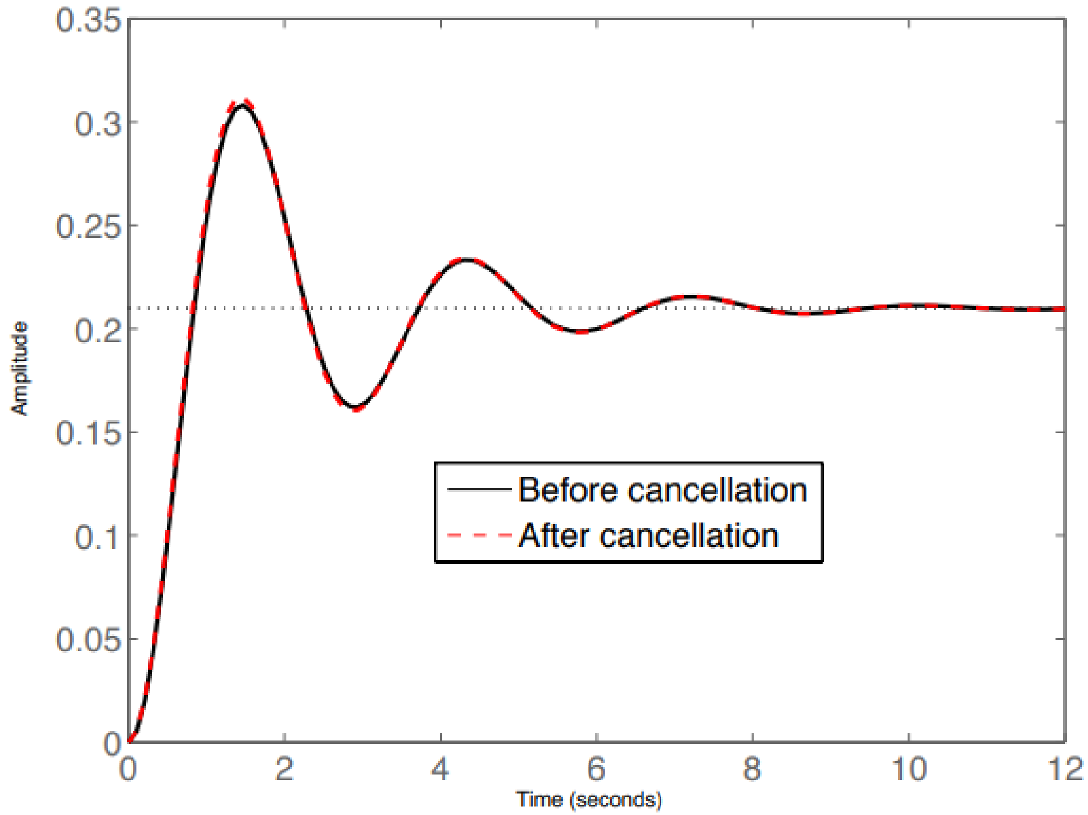
continued...

(c) $$G(s) = \frac{s+2.1}{(s+2)(s^2+2s+5)}$$

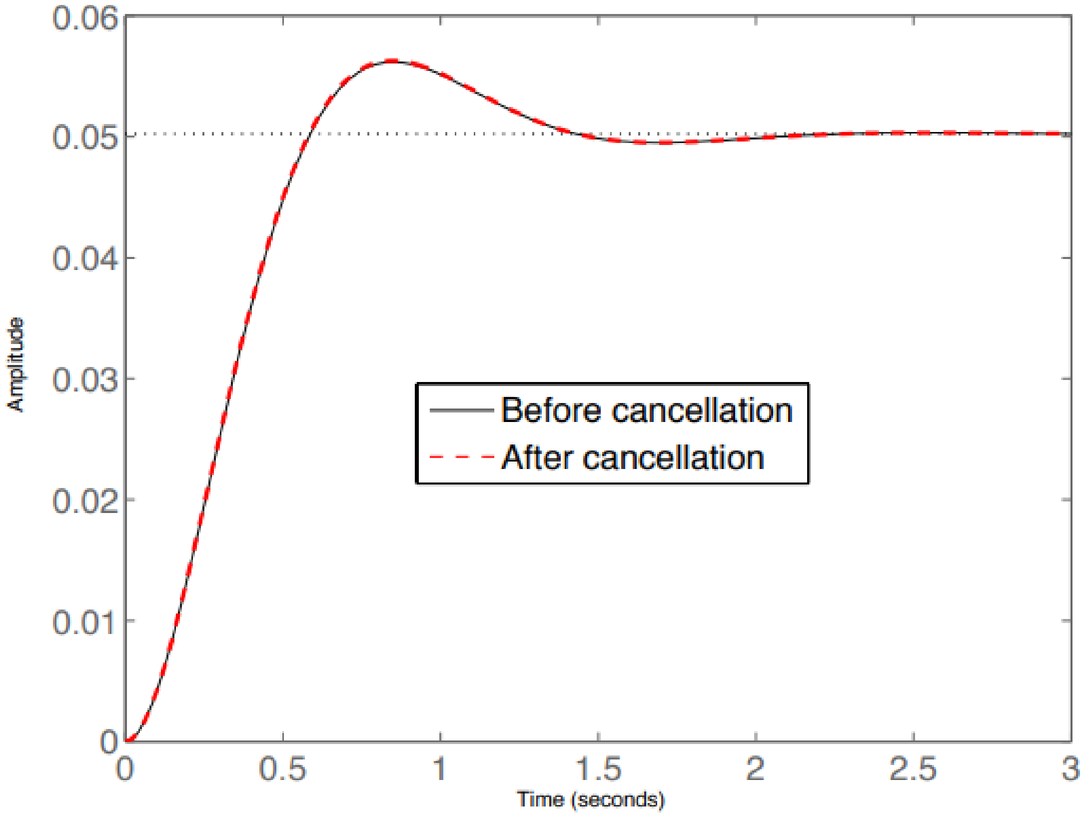
(d) $$G(s) = \frac{s+2.01}{(s+2)(s^2+5s+20)}$$

$$G_c(s) = \frac{1.05}{s^2+2s+5}$$

$$G_c(s) = \frac{1.005}{s^2+5s+20}$$



(c) Can definitely cancel here; almost no difference.



(d) Pole & zero completely cancel (almost).

Example

Determine the validity of a 2^{nd}-order step-response approximation for each transfer function shown below:

a. $$G(s) = \frac{185.71(s+7)}{(s+6.5)(s+10)(s+20)}$$

b. $$G(s) = \frac{197.14(s+7)}{(s+6.9)(s+10)(s+20)}$$

a. $$C(s) = \frac{1}{s} + \frac{0.8942}{s+20} - \frac{1.5918}{s+10} - \frac{0.3023}{s+6.5}$$

second-order approximation is not valid.

b. $$C(s) = \frac{1}{s} + \frac{0.9782}{s+20} - \frac{1.9078}{s+10} - \frac{0.0704}{s+6.5}$$

second-order approximation is valid.

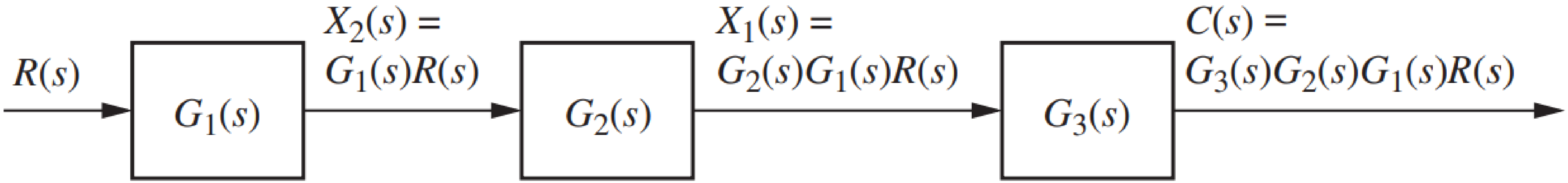
Control System

EEE 401 (Part 2)

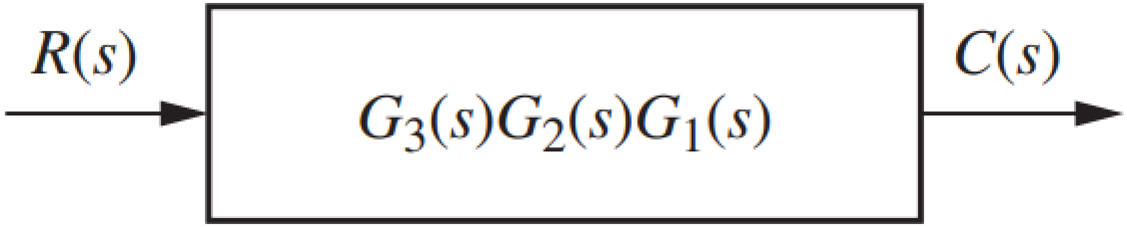
Dept. of EEE, BUBT

Block Diagram Topologies

(1) Cascade Form

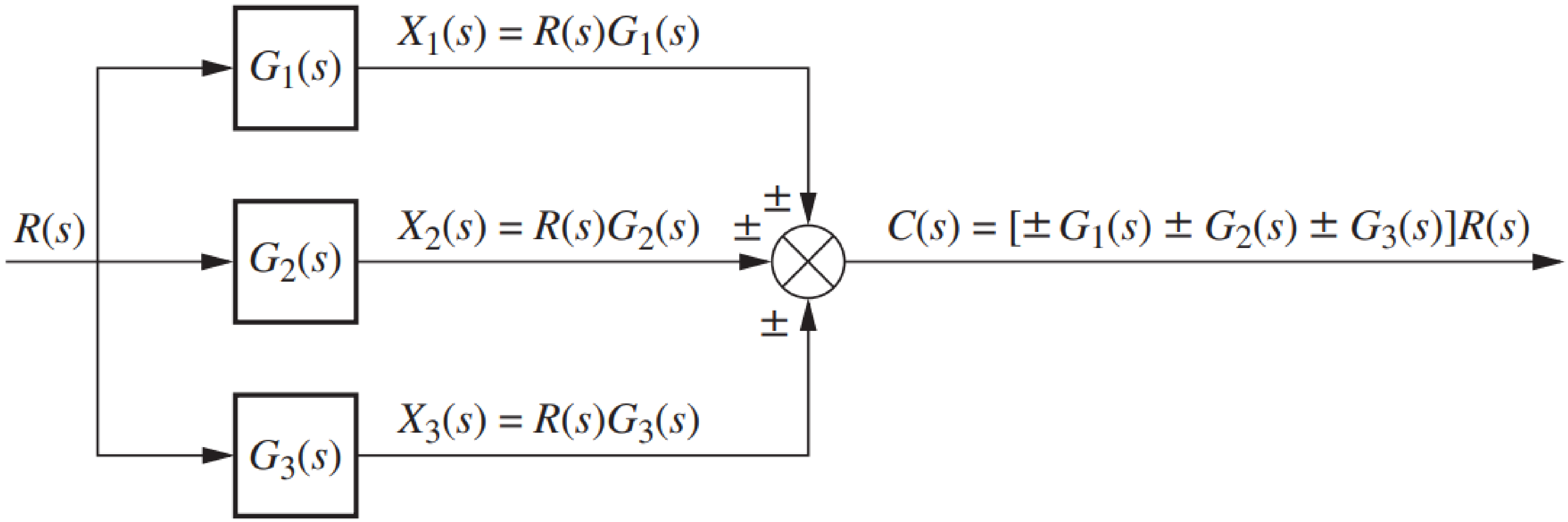


$$G_e(s) = G_3(s)G_2(s)G_1(s)$$

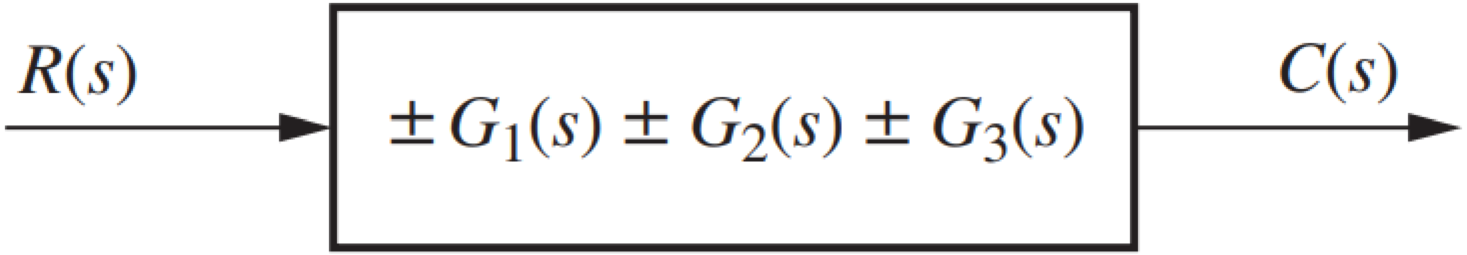


continued…

(2) Parallel Form

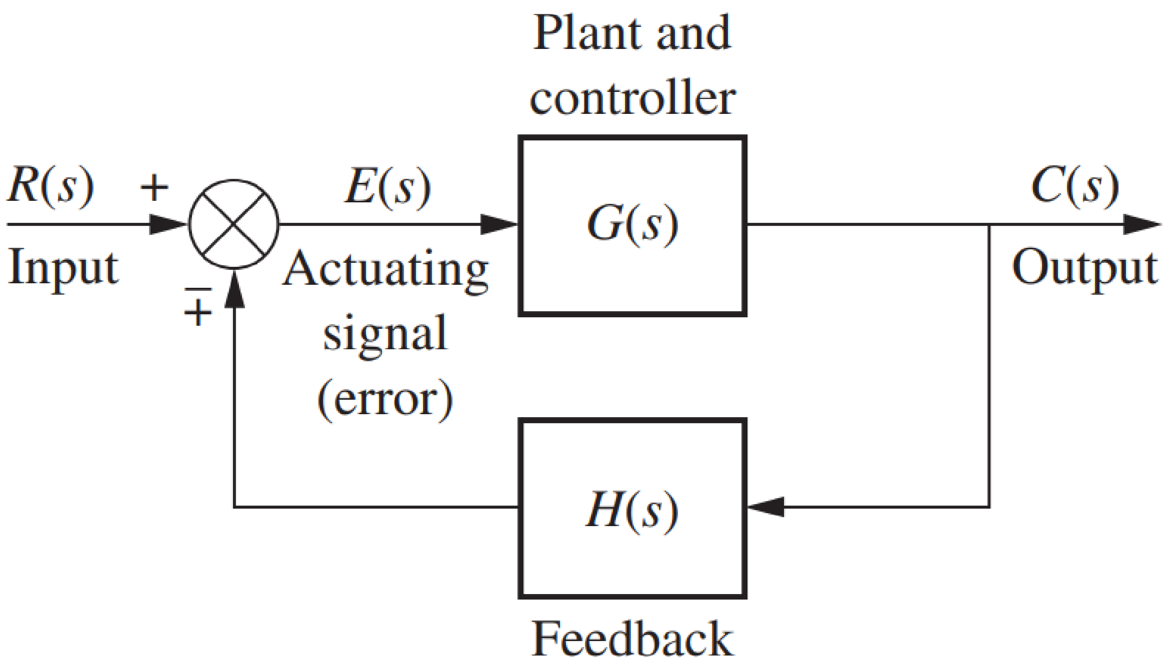


$$G_e(s) = \pm G_1(s) \pm G_2(s) \pm G_3(s)$$

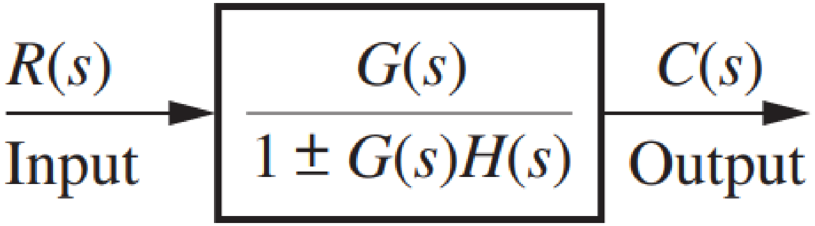


continued…

(3) Feedback Form

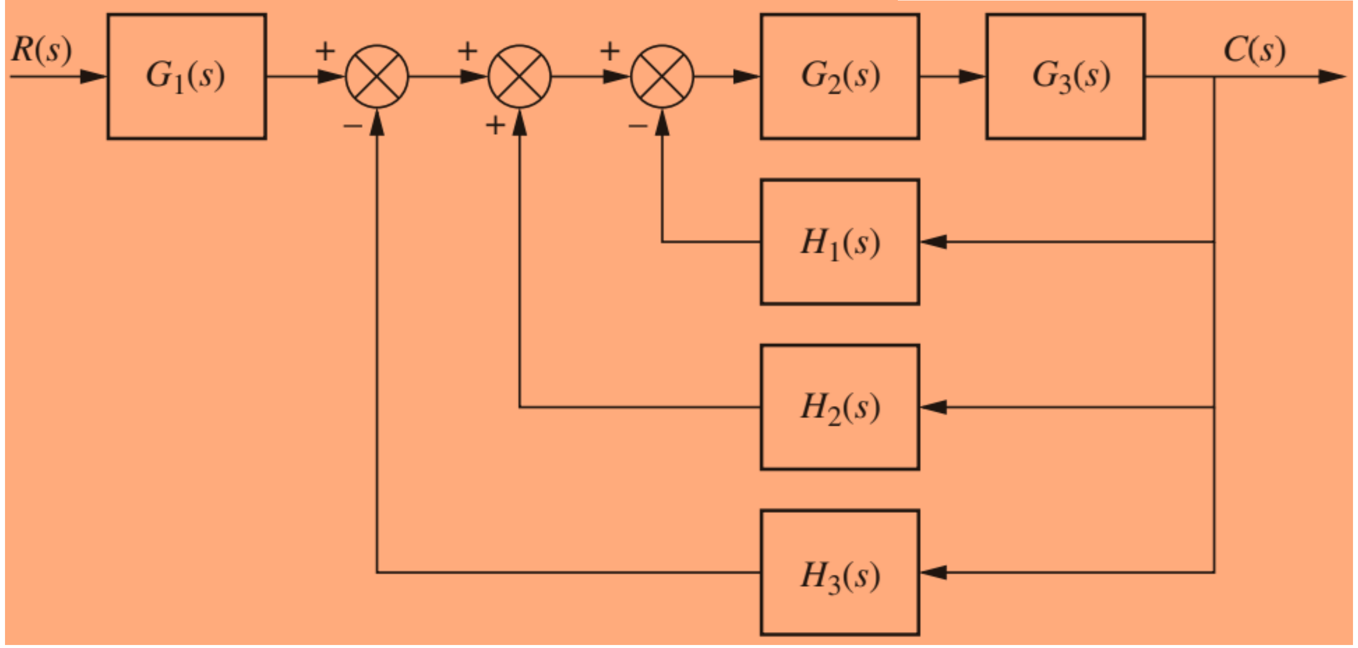


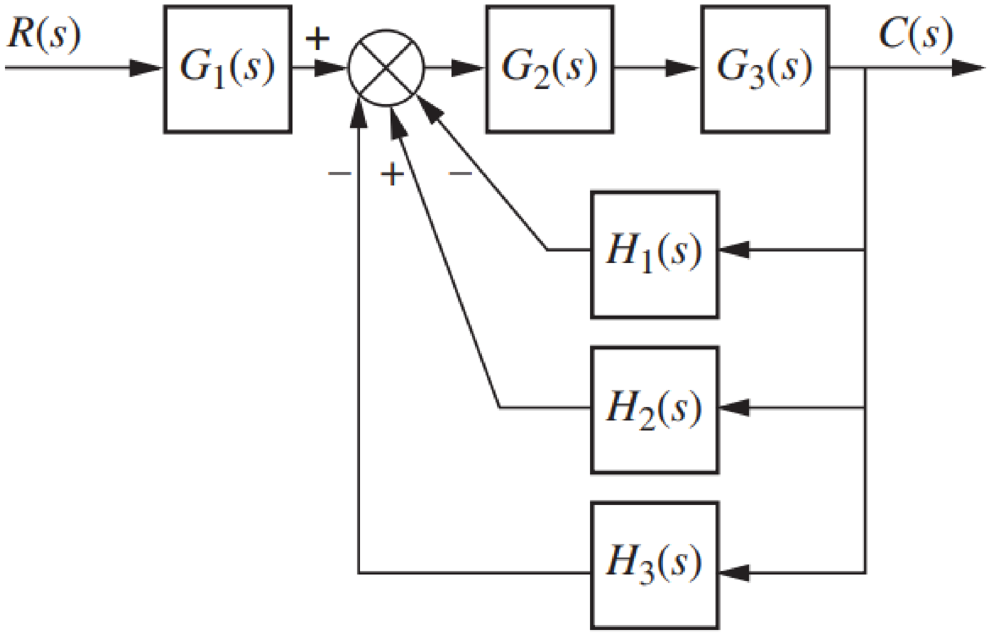
$$G_e(s) = \frac{G(s)}{1 \pm G(s)H(s)}$$

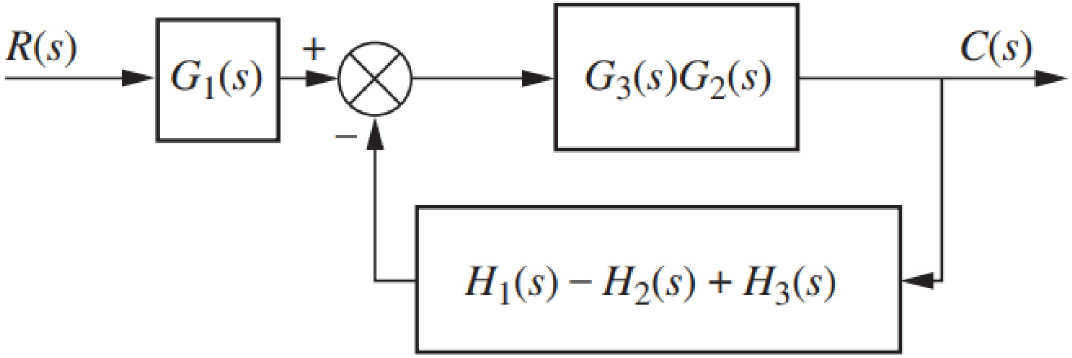


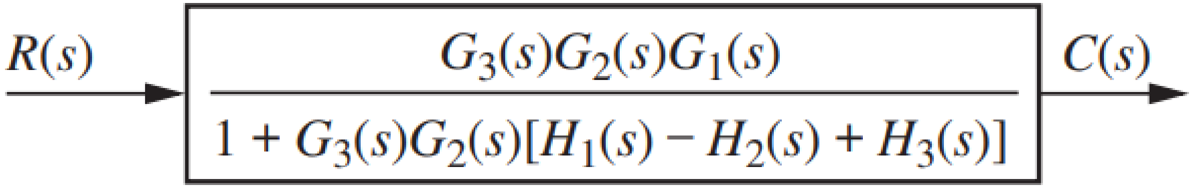
Example

Reduce the block diagram shown



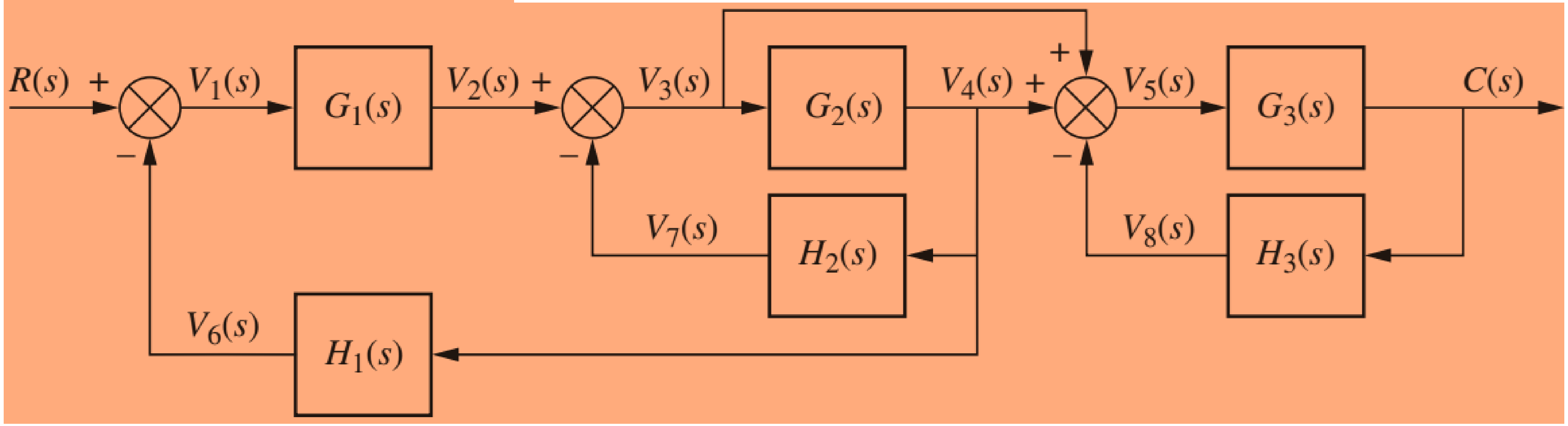


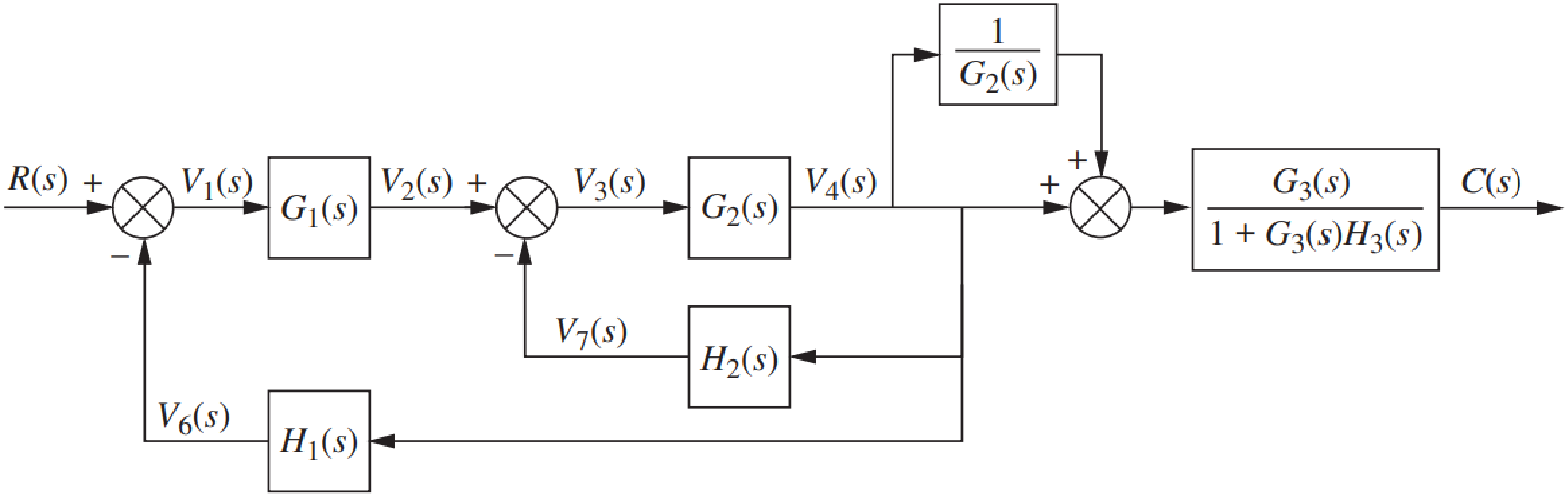




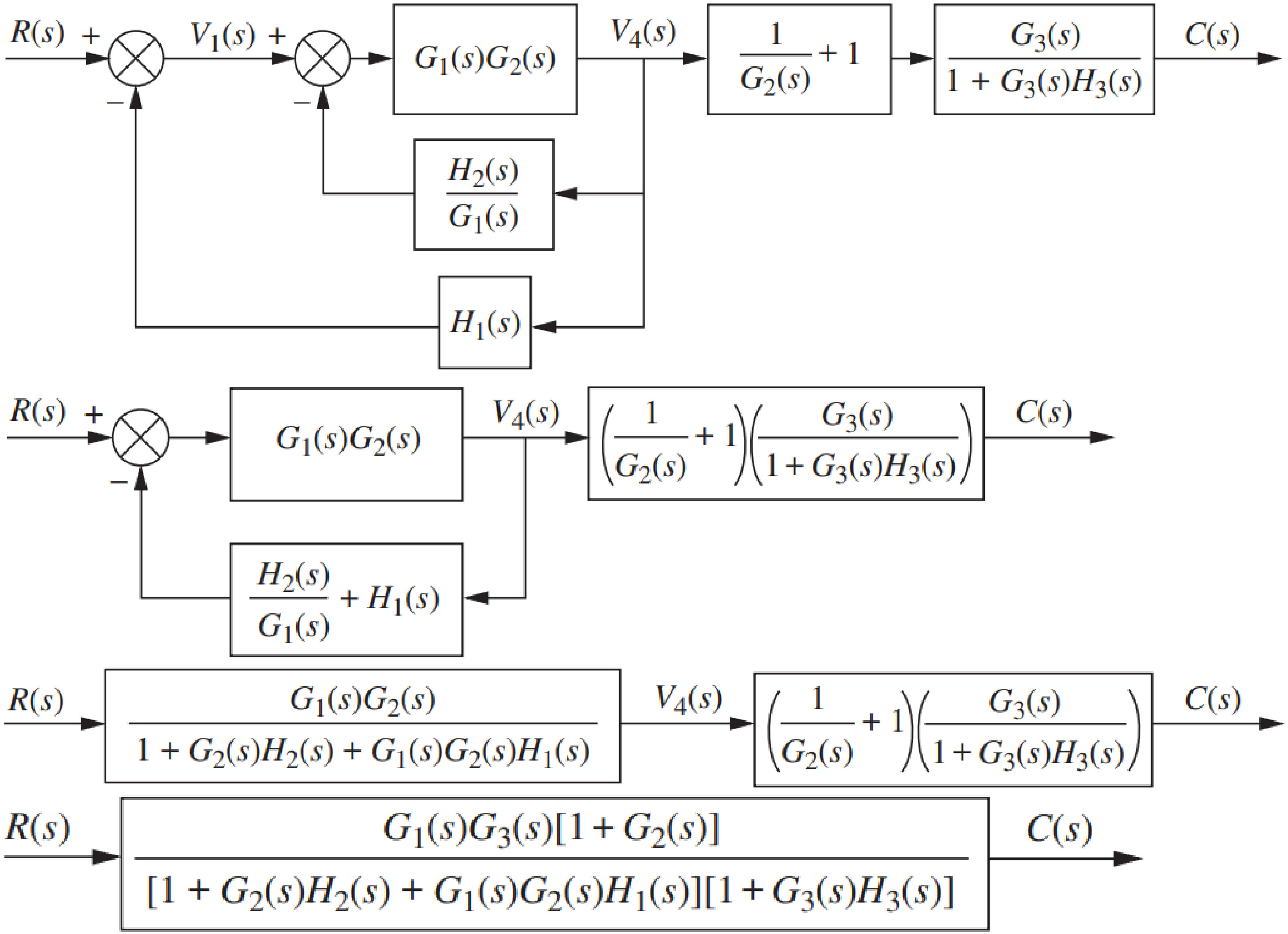
continued…

Reduce the system shown





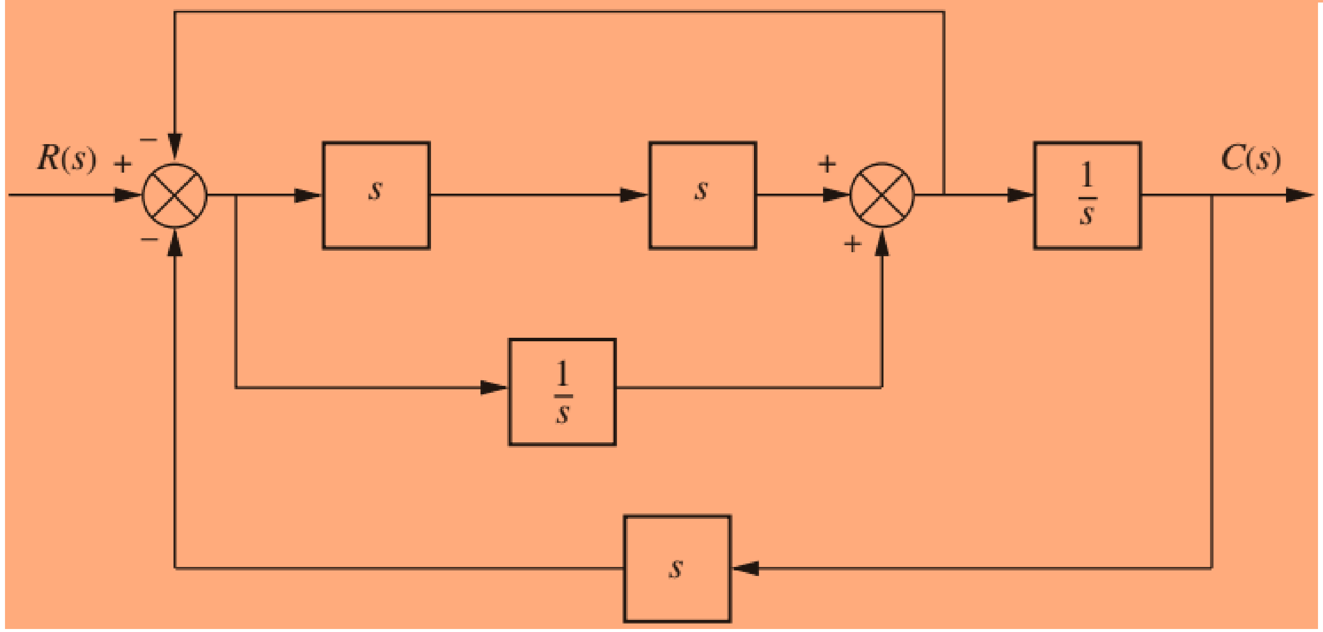
continued...

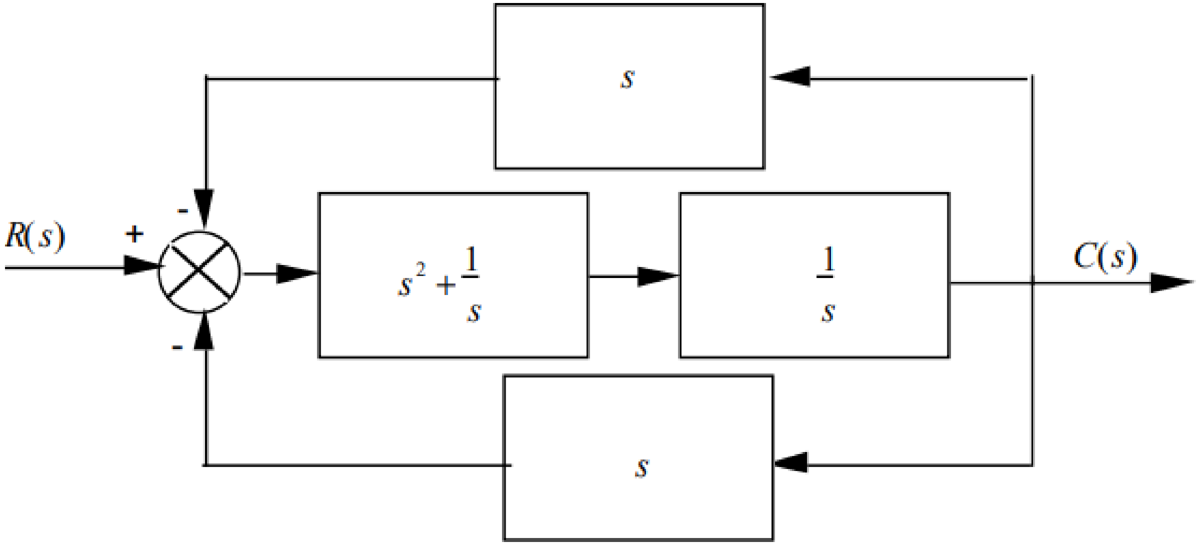


R(s)
+
V1(s)
+
−
−
G1(s)G2(s)
V4(s)
1/G2(s) + 1
G3(s) / 1 + G3(s)H3(s)
C(s)
H2(s)/G1(s)
H1(s)
R(s)
+
−
G1(s)G2(s)
V4(s)
(1/G2(s) + 1)(G3(s) / 1 + G3(s)H3(s))
C(s)
H2(s)/G1(s) + H1(s)
R(s)
G1(s)G2(s) / 1 + G2(s)H2(s) + G1(s)G2(s)H1(s)
V4(s)
(1/G2(s) + 1)(G3(s) / 1 + G3(s)H3(s))
C(s)
R(s)
G1(s)G3(s)[1 + G2(s)] / [1 + G2(s)H2(s) + G1(s)G2(s)H1(s)][1 + G3(s)H3(s)]
C(s)

continued…

Find the equivalent transfer function, $T(s) = C(s)/R(s)$, for the system





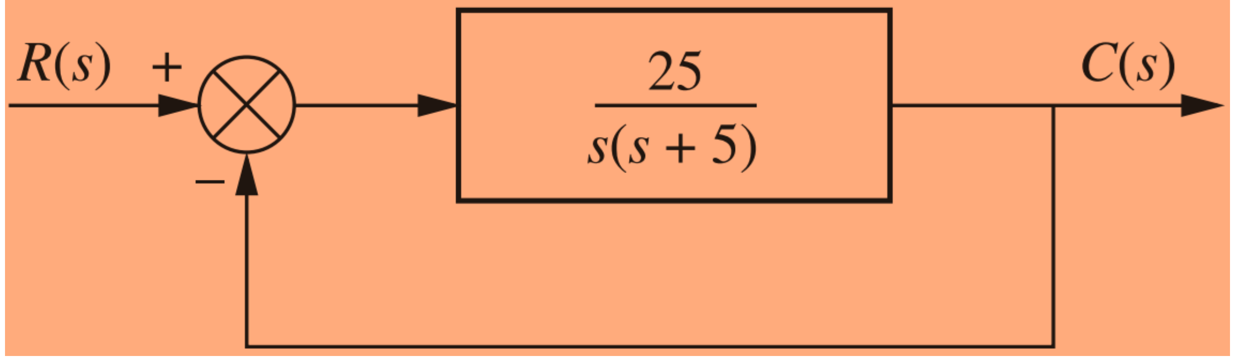
$$T(s) = \frac{\left(s^2 + \frac{1}{s}\right)\frac{1}{s}}{1 + \left(\left(s^2 + \frac{1}{s}\right)\frac{1}{s}\right)2s}$$

$$= \frac{\frac{s^3 + 1}{s^2}}{1 + \frac{s^3 + 1}{s^2} \cdot 2s}$$

$$= \frac{s^3 + 1}{2s^4 + s^2 + 2s}$$

continued…

For the system shown find the peak time, percent overshoot, and settling time.



$$T(s) = \frac{25}{s^2 + 5s + 25}$$

$$\omega_n = \sqrt{25} = 5$$

$$2\zeta\omega_n = 5 \rightarrow \zeta = 0.5$$

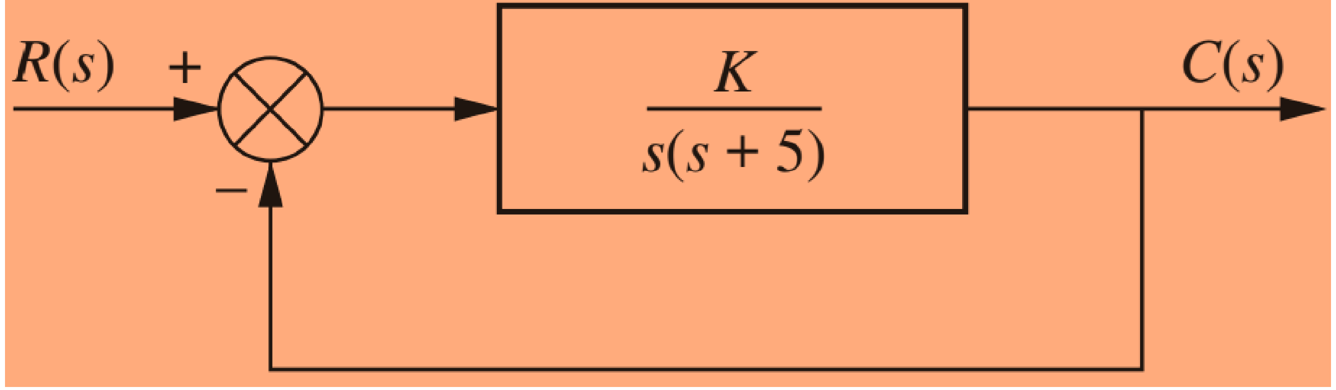
$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = 0.726 \text{ second}$$

$$\%OS = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \times 100 = 16.303$$

$$T_s = \frac{4}{\zeta\omega_n} = 1.6 \text{ seconds}$$

continued...

Design the value of gain. K, for the feedback control system so that the system will respond with a 10% overshoot.



$$T(s) = \frac{K}{s^2 + 5s + K}$$

$$\omega_n = \sqrt{K}$$

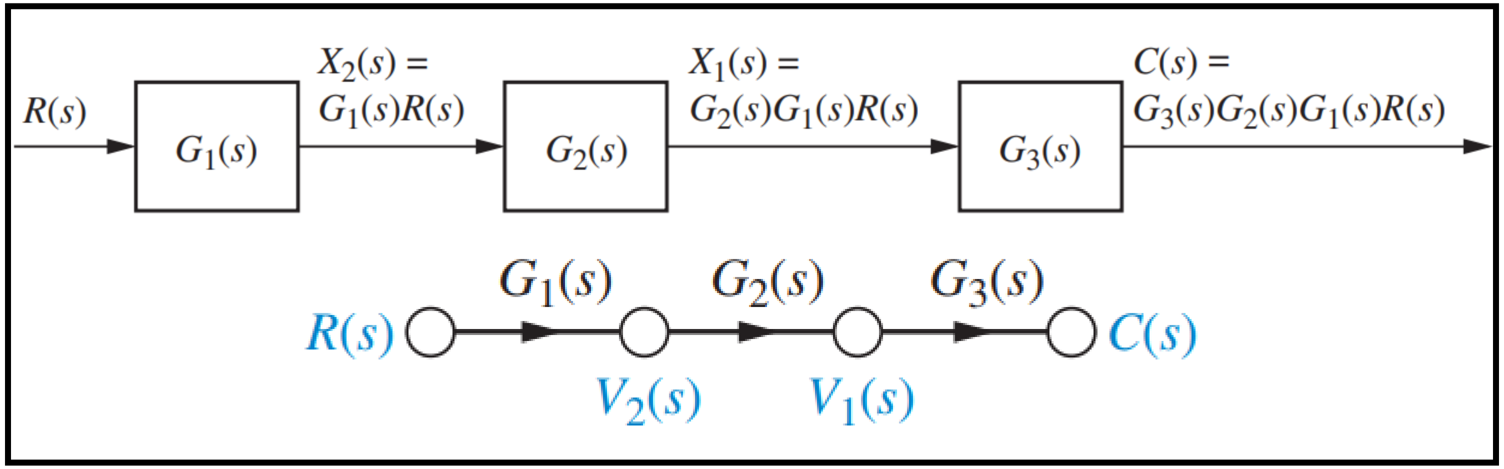
$$2\zeta\omega_n = 5 \rightarrow \zeta = \frac{5}{2\sqrt{K}}$$

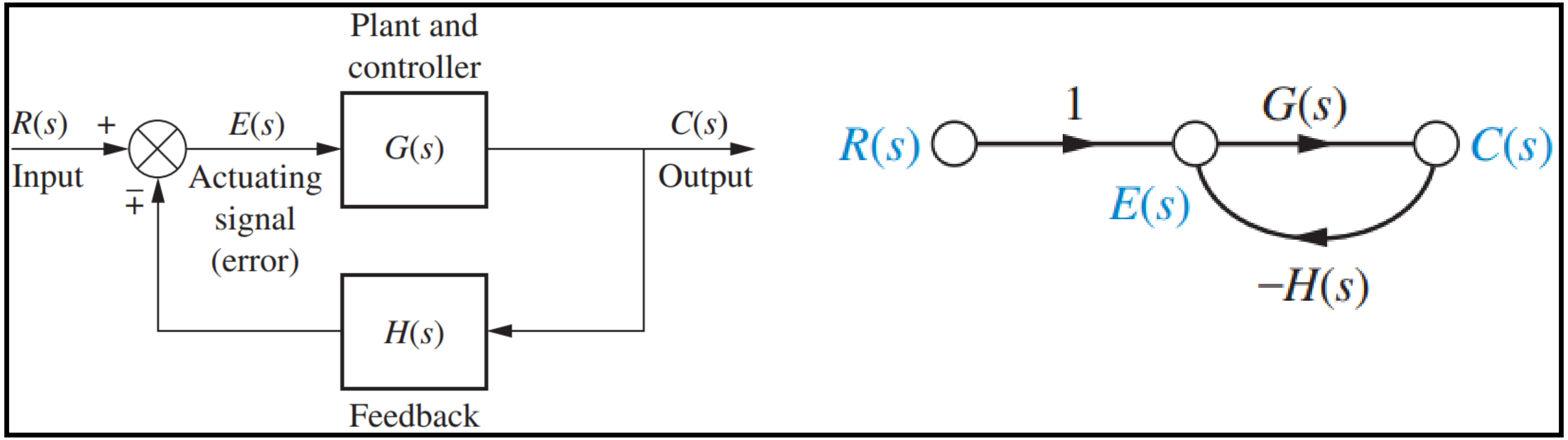
10% overshoot implies that $\zeta = 0.591$

$$K = 17.9$$

Signal-Flow Graphs

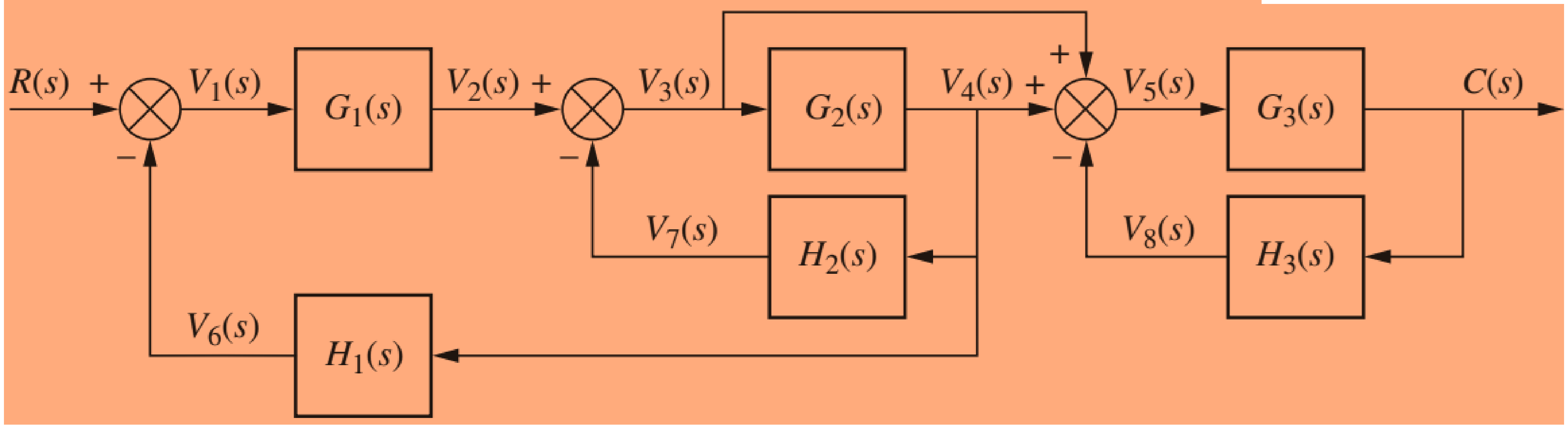
- Alternative to block diagrams

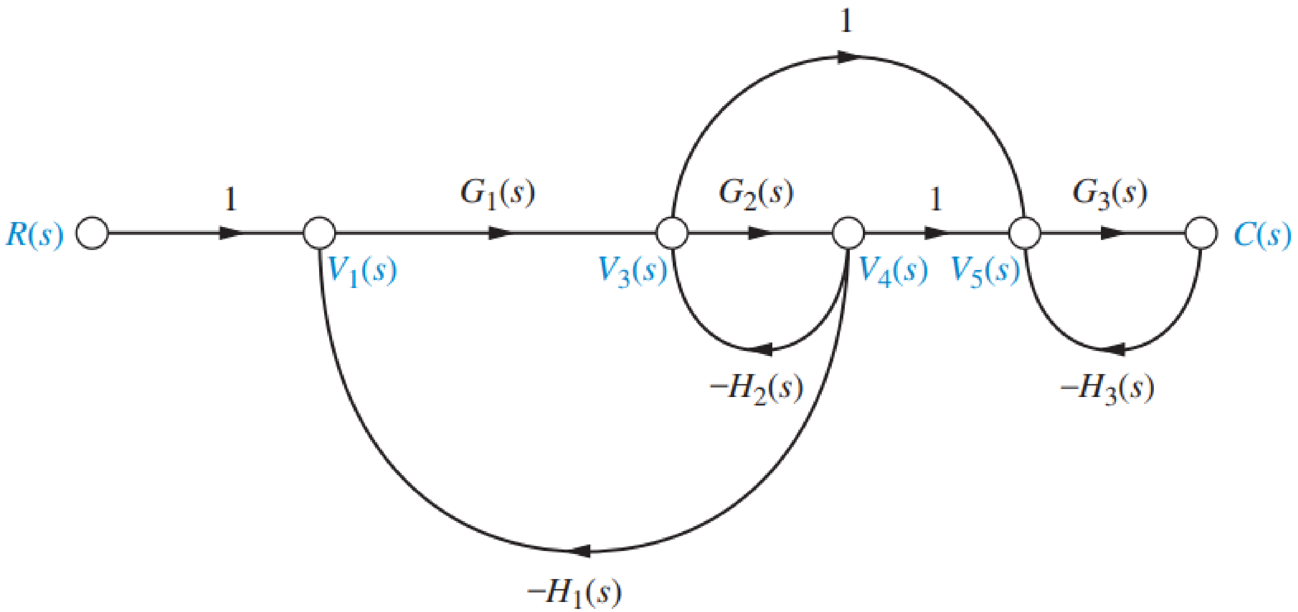




Example

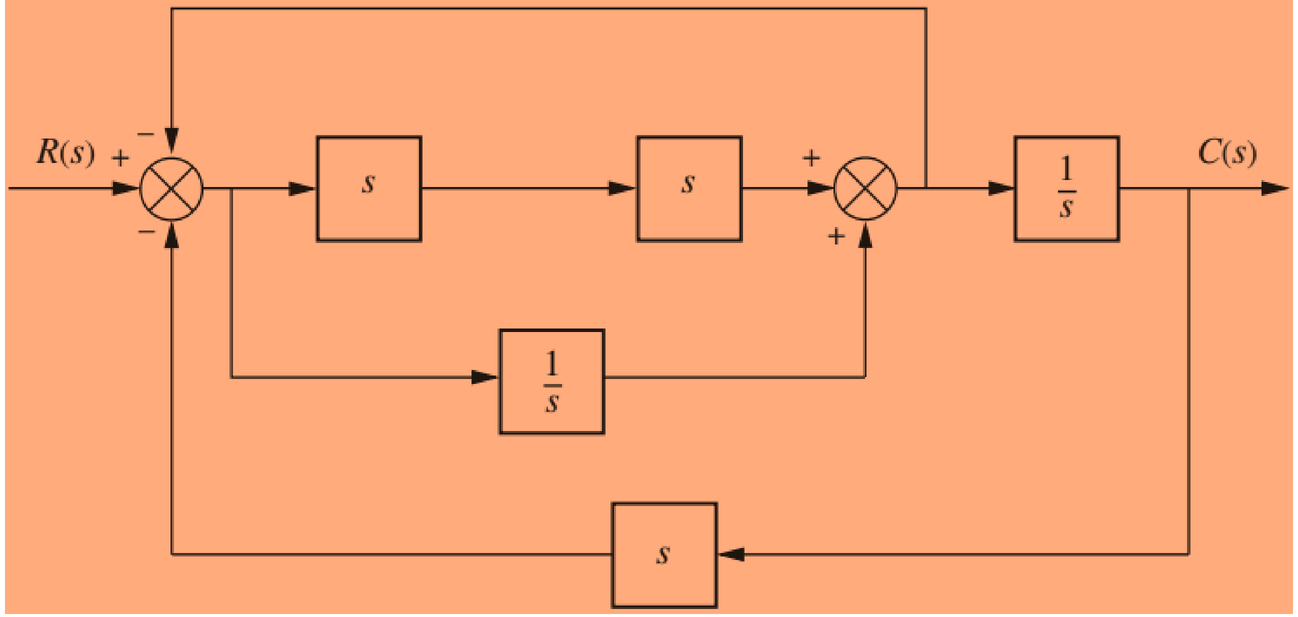
Convert the block diagram to a signal-flow graph.

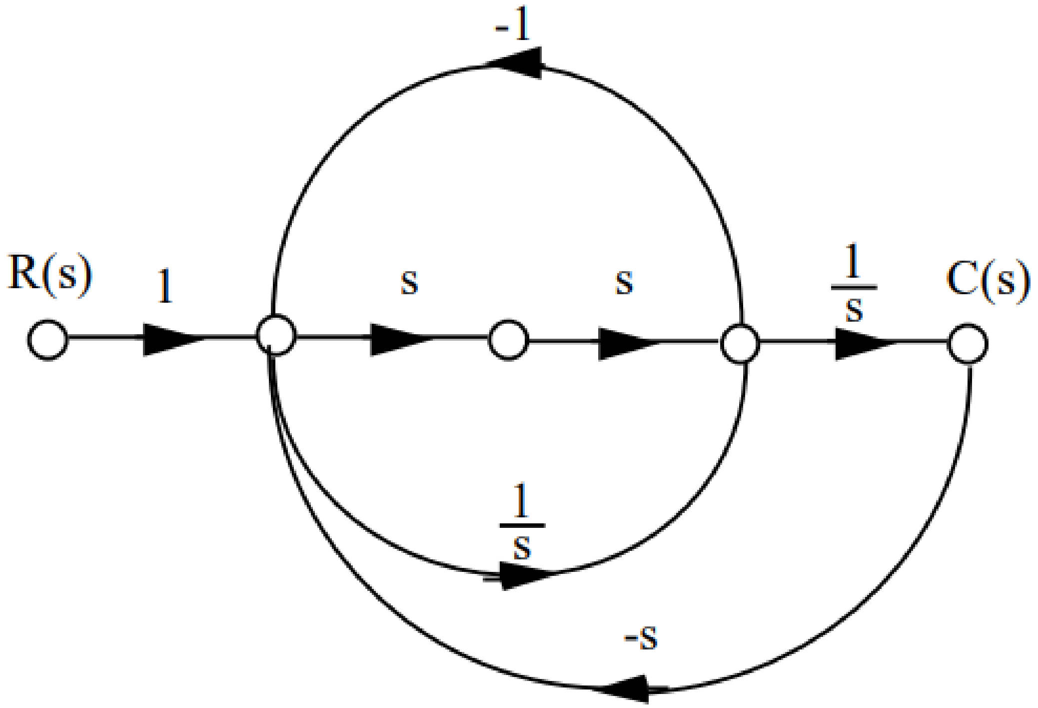




continued…

Convert the block diagram to a signal-flow graph.





Mason's Rule

$$G(s) = \frac{C(s)}{R(s)} = \frac{\sum_k T_k \Delta_k}{\Delta}$$

k = number of forward paths

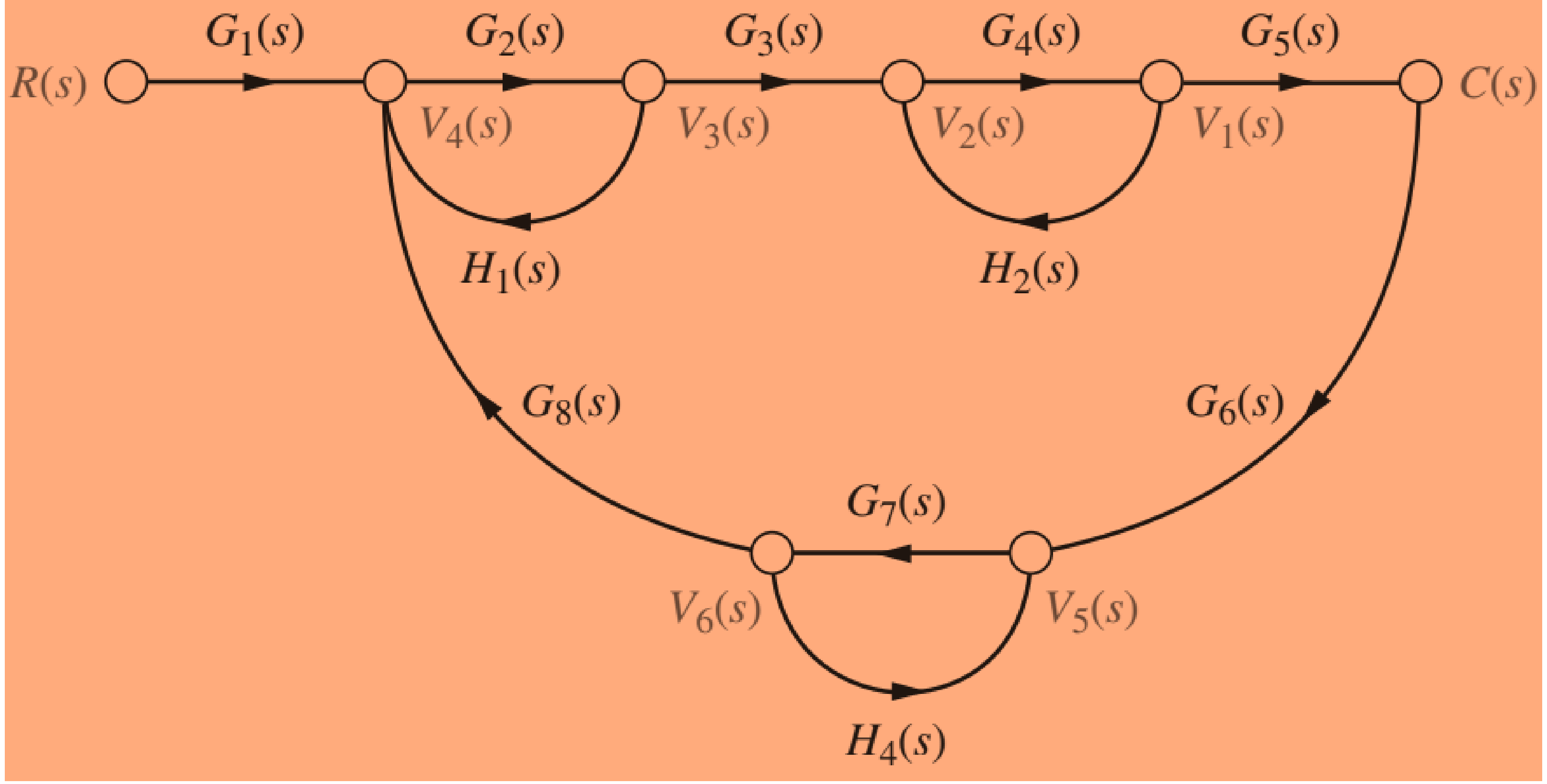
T_k = the kth forward-path gain

Δ = 1 − Σ loop gains + Σ nontouching-loop gains taken two at a time − Σ nontouching-loop gains taken three at a time + Σ nontouching-loop gains taken four at a time − . . .

Δ_k = Δ − Σ loop gain terms in Δ that touch the kth forward path

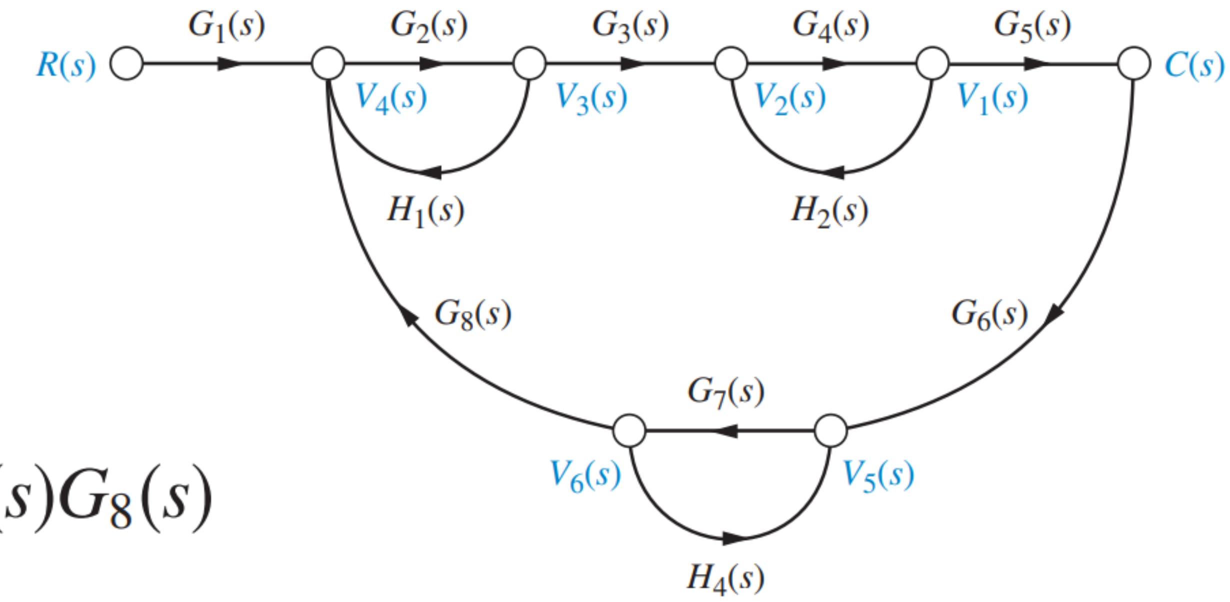
Example

Find the transfer function, $C(s)/R(s)$, for the signal-flow graph



forward-path gains $G_1(s)G_2(s)G_3(s)G_4(s)G_5(s)$

continued...



loop gains

1. $G_2(s)H_1(s)$
2. $G_4(s)H_2(s)$
3. $G_7(s)H_4(s)$
4. $G_2(s)G_3(s)G_4(s)G_5(s)G_6(s)G_7(s)G_8(s)$

nontouching loops taken two at a time

Loop 1 and loop 2: $G_2(s)H_1(s)G_4(s)H_2(s)$

Loop 1 and loop 3: $G_2(s)H_1(s)G_7(s)H_4(s)$

Loop 2 and loop 3: $G_4(s)H_2(s)G_7(s)H_4(s)$

Non-touching loops are those not sharing any common **node** or **branch**.

continued…

nontouching loops taken three at a time

Loops 1, 2, and 3: $G_2(s)H_1(s)G_4(s)H_2(s)G_7(s)H_4(s)$

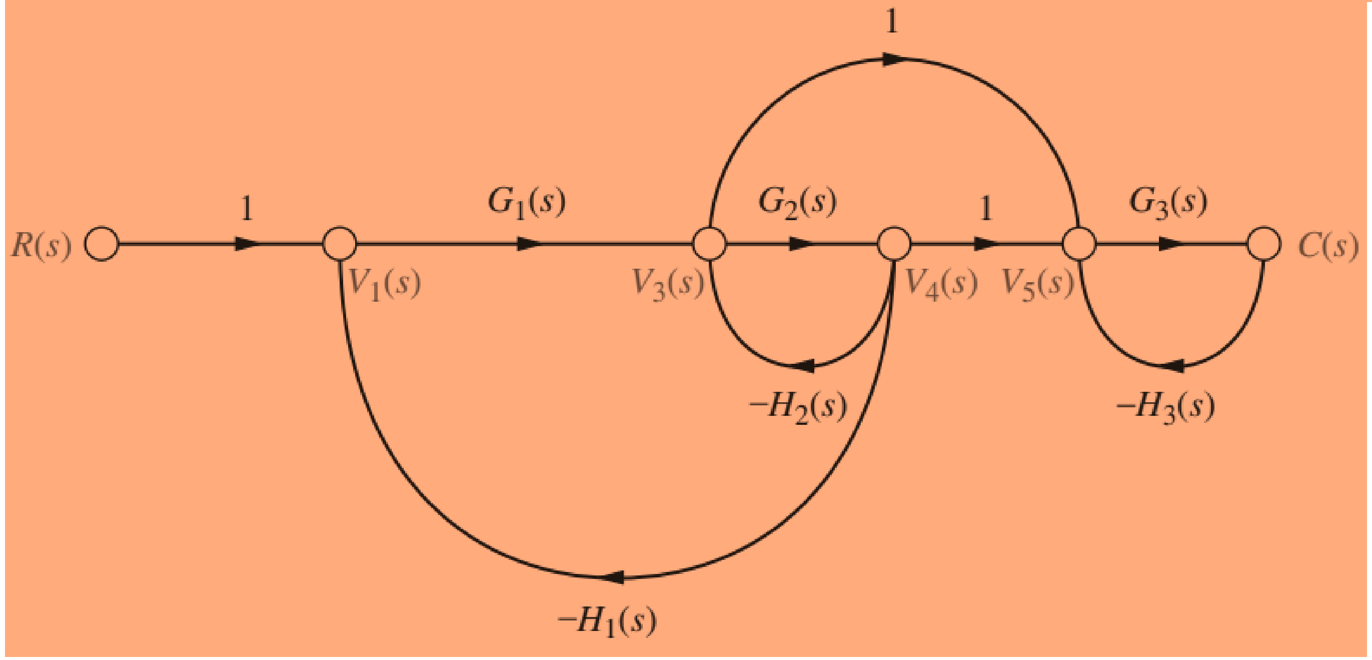
$$\Delta = 1 - [G_2(s)H_1(s) + G_4(s)H_2(s) + G_7(s)H_4(s) + G_2(s)G_3(s)G_4(s)G_5(s)G_6(s)G_7(s)G_8(s)] + [G_2(s)H_1(s)G_4(s)H_2(s) + G_2(s)H_1(s)G_7(s)H_4(s) + G_4(s)H_2(s)G_7(s)H_4(s)] - [G_2(s)H_1(s)G_4(s)H_2(s)G_7(s)H_4(s)]$$

Since there is only one forward path, $\Delta_1 = 1 - G_7(s)H_4(s)$

$$G(s) = \frac{T_1\Delta_1}{\Delta} = \frac{[G_1(s)G_2(s)G_3(s)G_4(s)G_5(s)][1 - G_7(s)H_4(s)]}{\Delta}$$

continued…

Find the transfer function, $C(s)/R(s)$, for the signal-flow graph



forward-path gains

$G_1G_2G_3$ and G_1G_3.

loop gains

$-G_1G_2H_1$, $-G_2H_2$, and $-G_3H_3$.

nontouching loops taken two at a time

$$[-G_1G_2H_1][-G_3H_3] = G_1G_2G_3H_1H_3$$

$$[-G_2H_2][-G_3H_3] = G_2G_3H_2H_3.$$

continued…

$$\Delta = 1 + G_1G_2H_1 + G_2H_2 + G_3H_3 + G_1G_2G_3H_1H_3 + G_2G_3H_2H_3.$$

$\Delta_1 = 1$ and $\Delta_2 = 1$

$$T(s) = \frac{C(s)}{R(s)} = \frac{\sum_k T_k\Delta_k}{\Delta} = \frac{T_1\Delta_1 + T_2\Delta_2}{\Delta}$$

$$= \frac{G_1G_2G_3 \cdot 1 + G_1G_3 \cdot 1}{\Delta}$$

$$= \frac{G_1G_2G_3 + G_1G_3}{1 + G_1G_2H_1 + G_2H_2 + G_3H_3 + G_1G_2G_3H_1H_3 + G_2G_3H_2H_3}$$

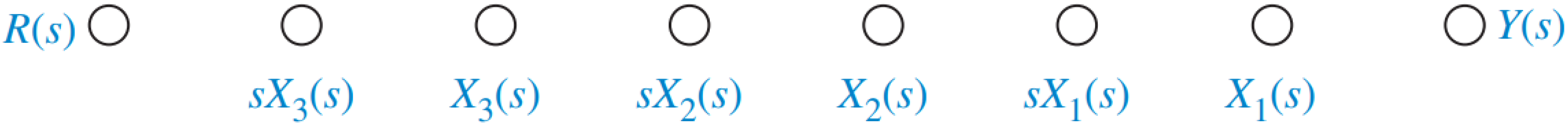
Signal-Flow Graphs of State Equations

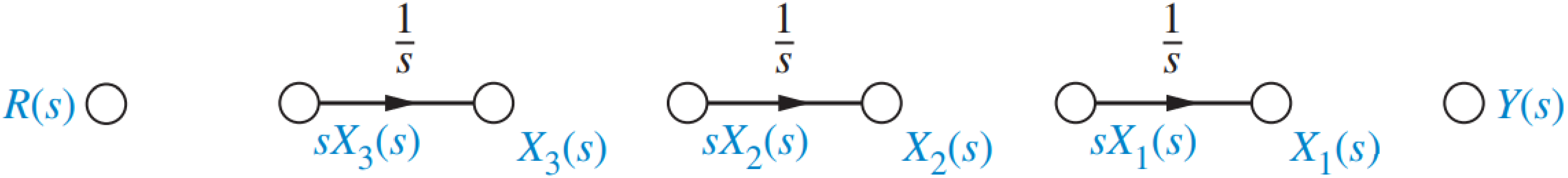
$$\dot{x}_1 = 2x_1 - 5x_2 + 3x_3 + 2r$$

$$\dot{x}_2 = -6x_1 - 2x_2 + 2x_3 + 5r$$

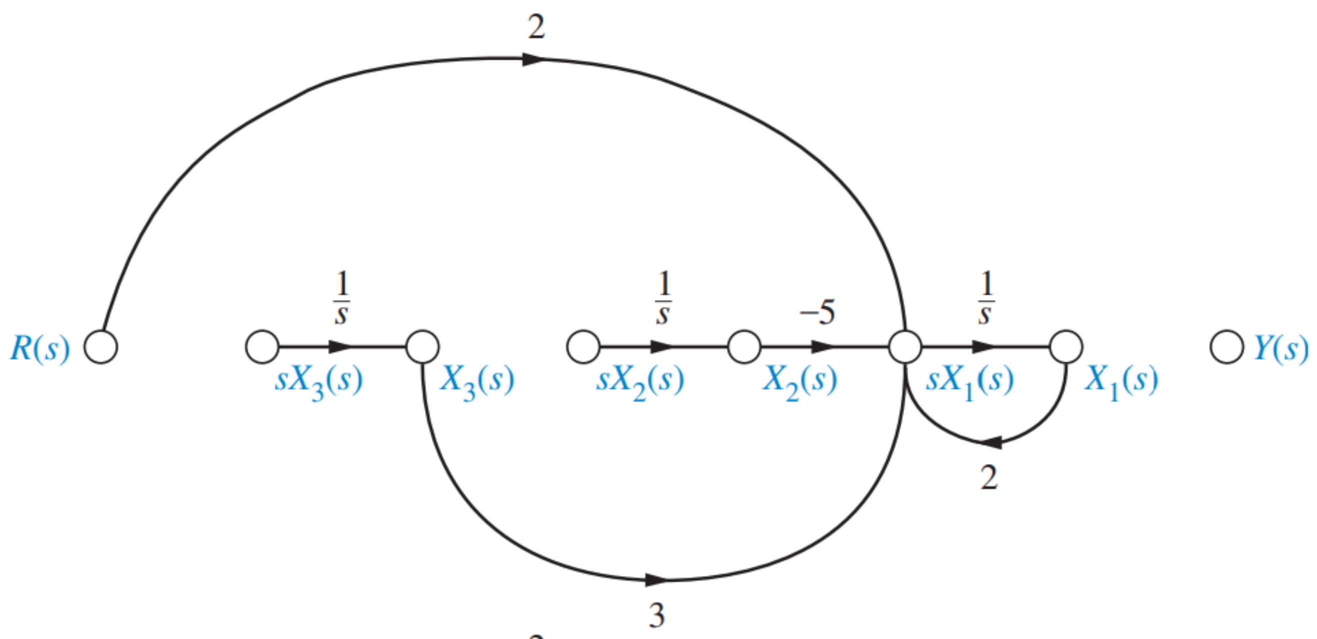
$$\dot{x}_3 = x_1 - 3x_2 - 4x_3 + 7r$$

$$y = -4x_1 + 6x_2 + 9x_3$$

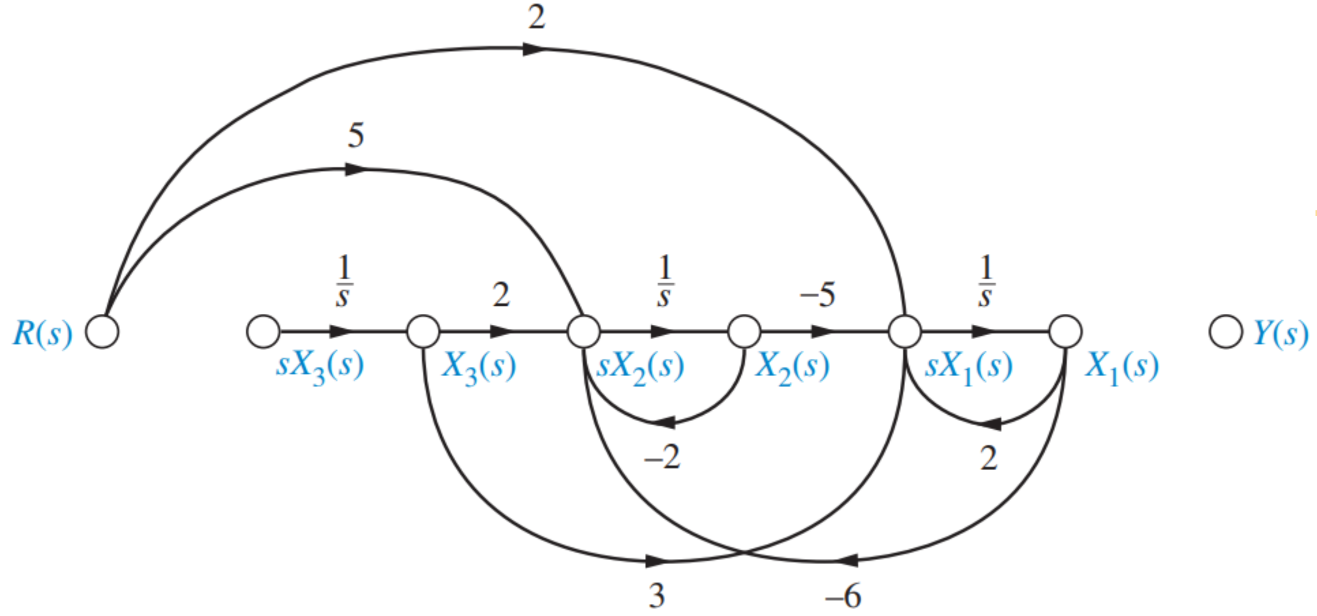




continued…

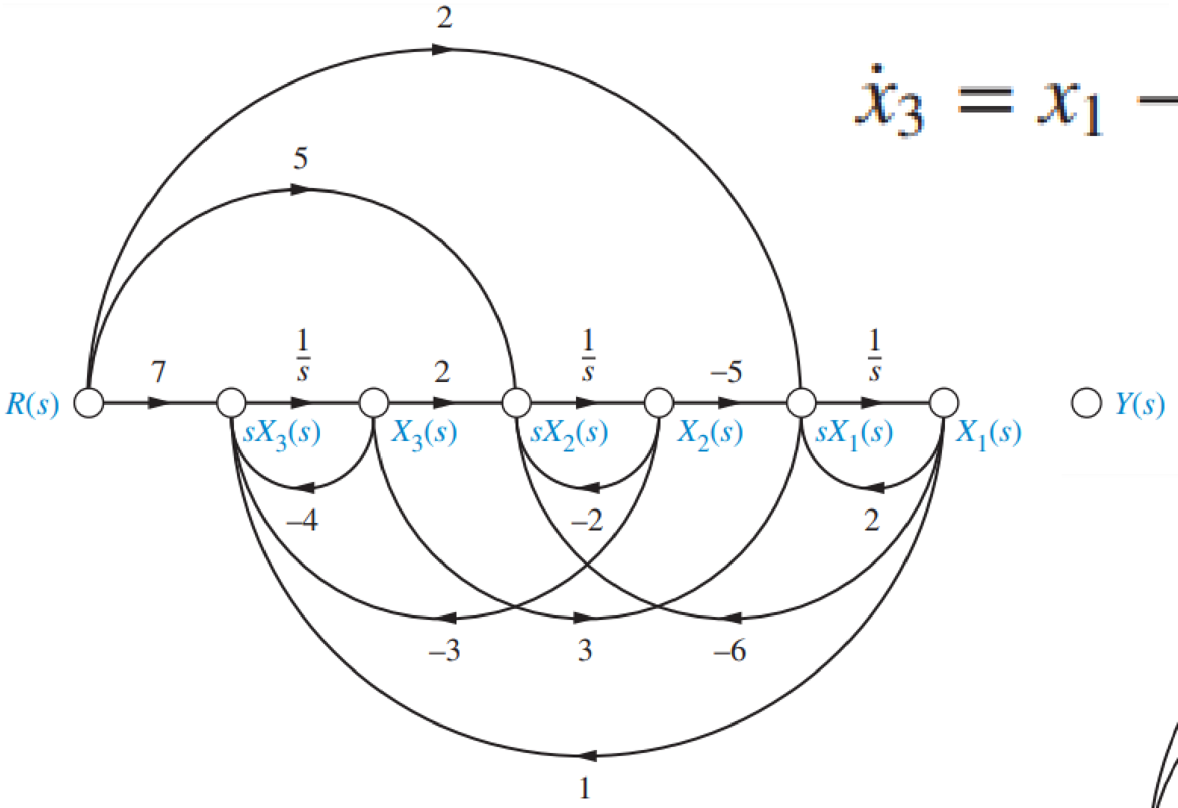


$$\dot{x}_1 = 2x_1 - 5x_2 + 3x_3 + 2r$$

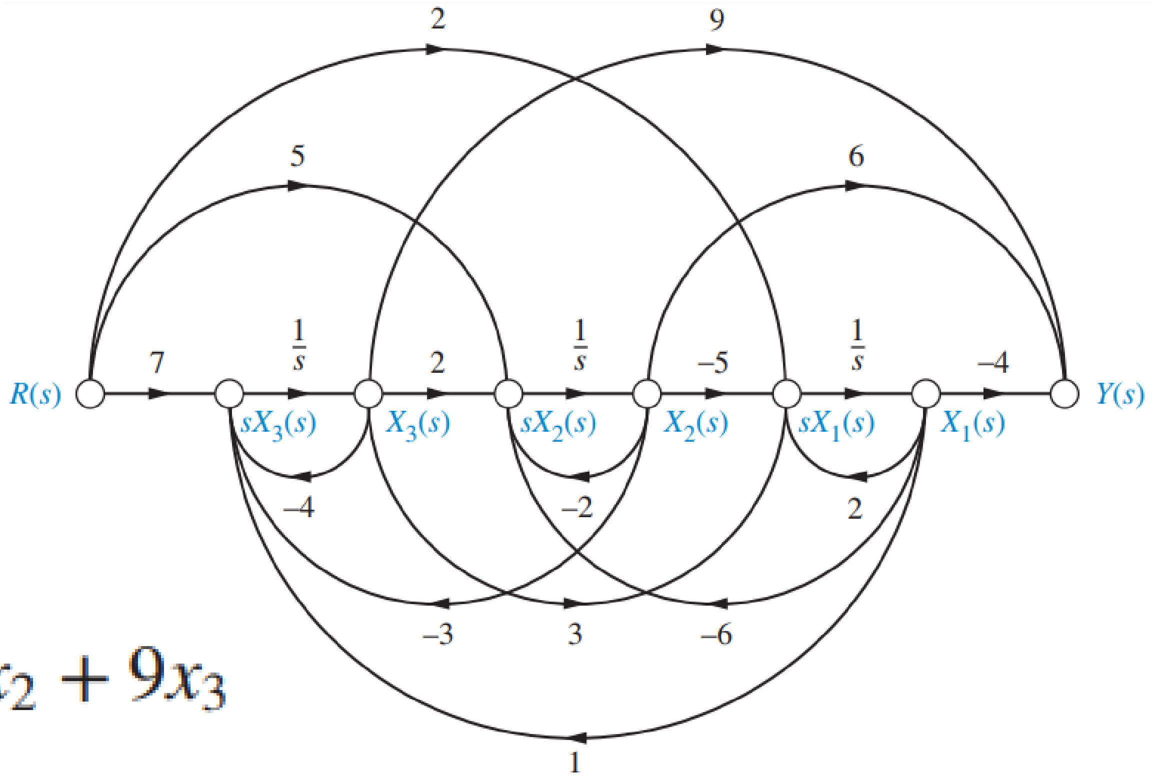


$$\dot{x}_2 = -6x_1 - 2x_2 + 2x_3 + 5r$$

continued...



$$\dot{x}_3 = x_1 - 3x_2 - 4x_3 + 7r$$



$$y = -4x_1 + 6x_2 + 9x_3$$

Example

Draw a signal-flow graph for the following state and output equations:

$$\dot{\mathbf{x}} = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -3 & 1 \\ -3 & -4 & -5 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r$$

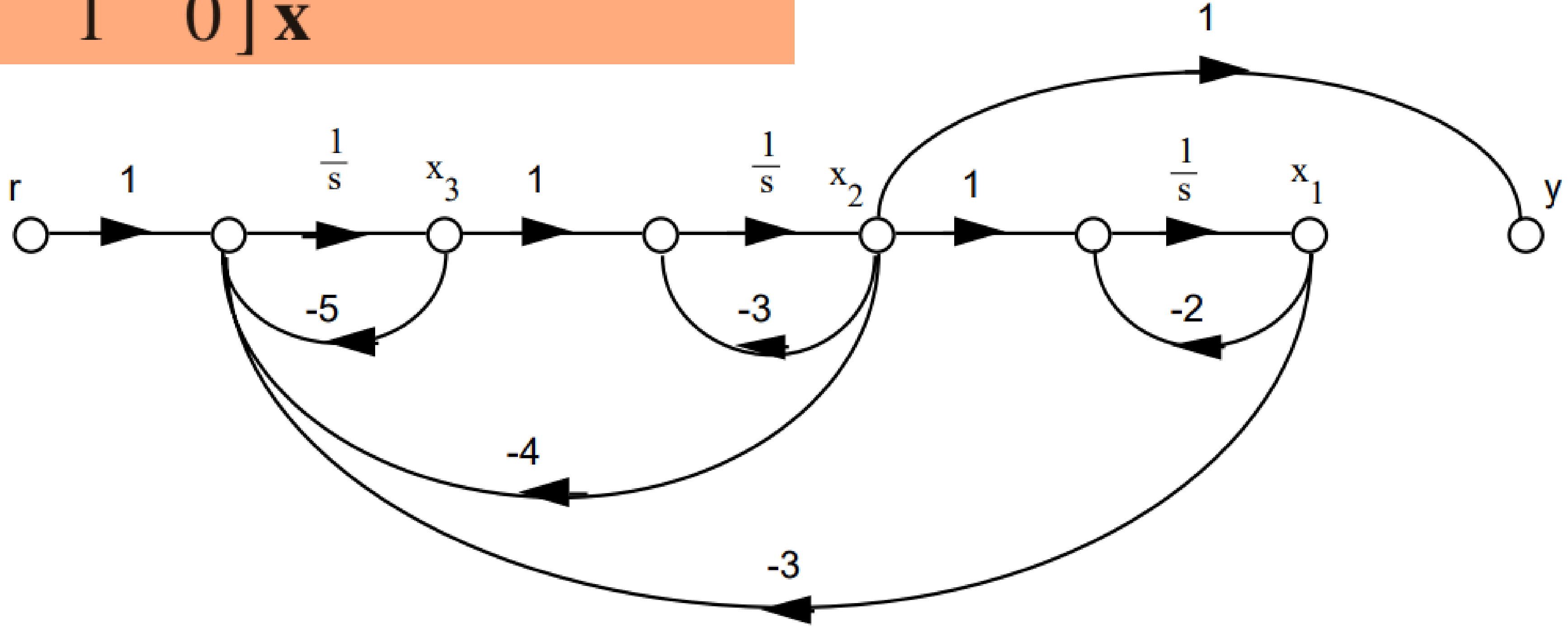
$$y = [0 \quad 1 \quad 0]\,\mathbf{x}$$

$$\dot{x}_1 = -2x_1 + x_2$$

$$\dot{x}_2 = -3x_2 + x_3$$

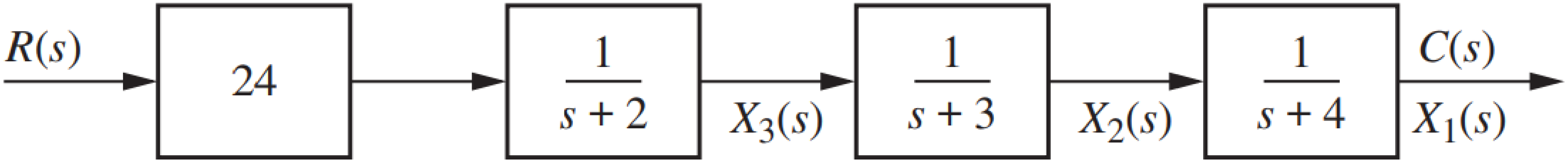
$$\dot{x}_3 = -3x_1 - 4x_2 - 5x_3 + r$$

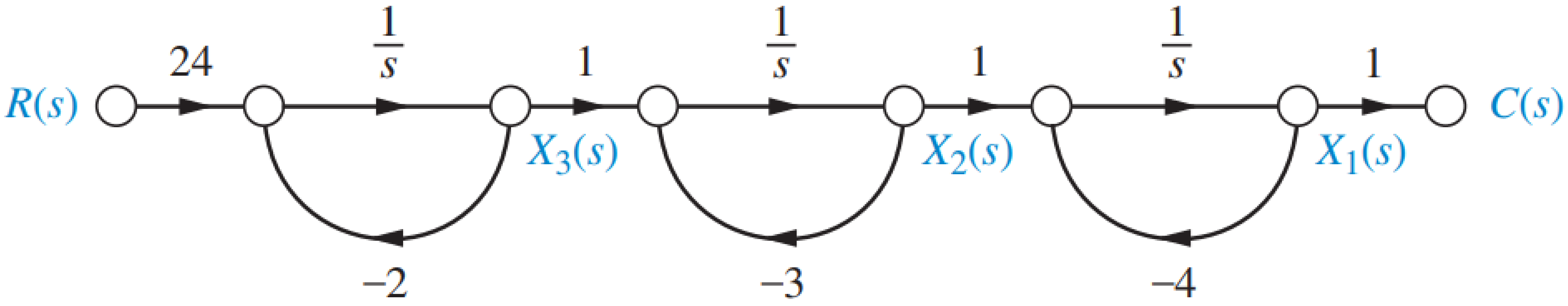
$$y = x_2$$



Alternate Representation
[1] Cascade Form

$$\frac{C(s)}{R(s)} = \frac{24}{(s+2)(s+3)(s+4)}$$



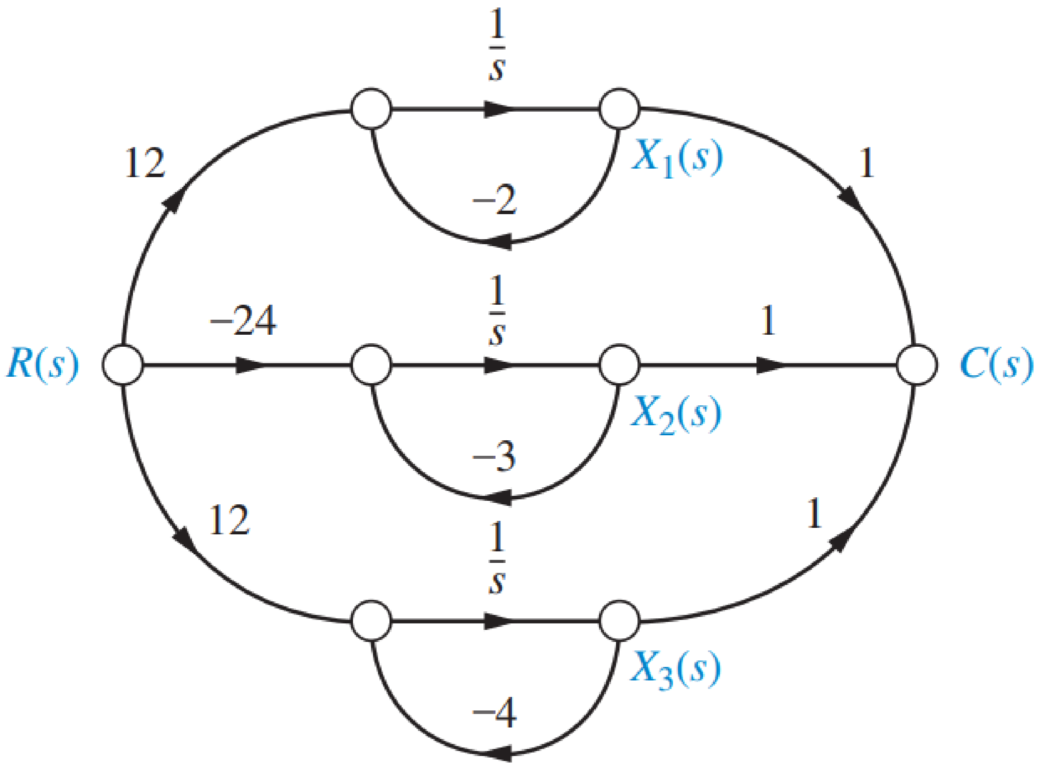


$$\dot{\mathbf{x}} = \begin{bmatrix} -4 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & -2 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 24 \end{bmatrix} r \qquad y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\mathbf{x}$$

Alternate Representation
[**2**] Parallel Form

$$\frac{C(s)}{R(s)} = \frac{24}{(s+2)(s+3)(s+4)} = \frac{12}{(s+2)} - \frac{24}{(s+3)} + \frac{12}{(s+4)}$$

$$C(s) = R(s)\frac{12}{(s+2)} - R(s)\frac{24}{(s+3)} + R(s)\frac{12}{(s+4)}$$



$$\dot{\mathbf{x}} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -4 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 12 \\ -24 \\ 12 \end{bmatrix} r$$

$$y = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}\mathbf{x}$$

Alternate Representation
[**3**] Controller Canonical Form

$$G(s) = \frac{C(s)}{R(s)} = \frac{s^2 + 7s + 2}{s^3 + 9s^2 + 26s + 24}$$

Phase Variable Form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r$$

$$y = \begin{bmatrix} 2 & 7 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Renumbering in Reverse Order

$$\begin{bmatrix} \dot{x}_3 \\ \dot{x}_2 \\ \dot{x}_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} \begin{bmatrix} x_3 \\ x_2 \\ x_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r$$

$$y = \begin{bmatrix} 2 & 7 & 1 \end{bmatrix} \begin{bmatrix} x_3 \\ x_2 \\ x_1 \end{bmatrix}$$

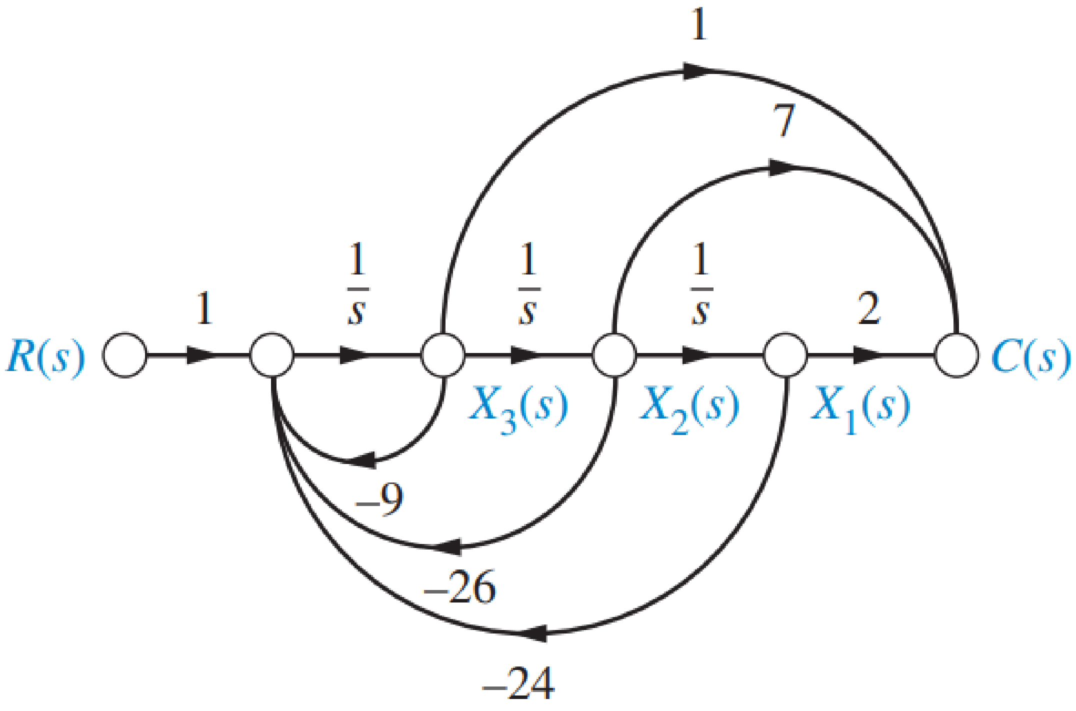
Reordering in Ascending Order

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -9 & -26 & -24 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} r$$

$$y = \begin{bmatrix} 1 & 7 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

continued...

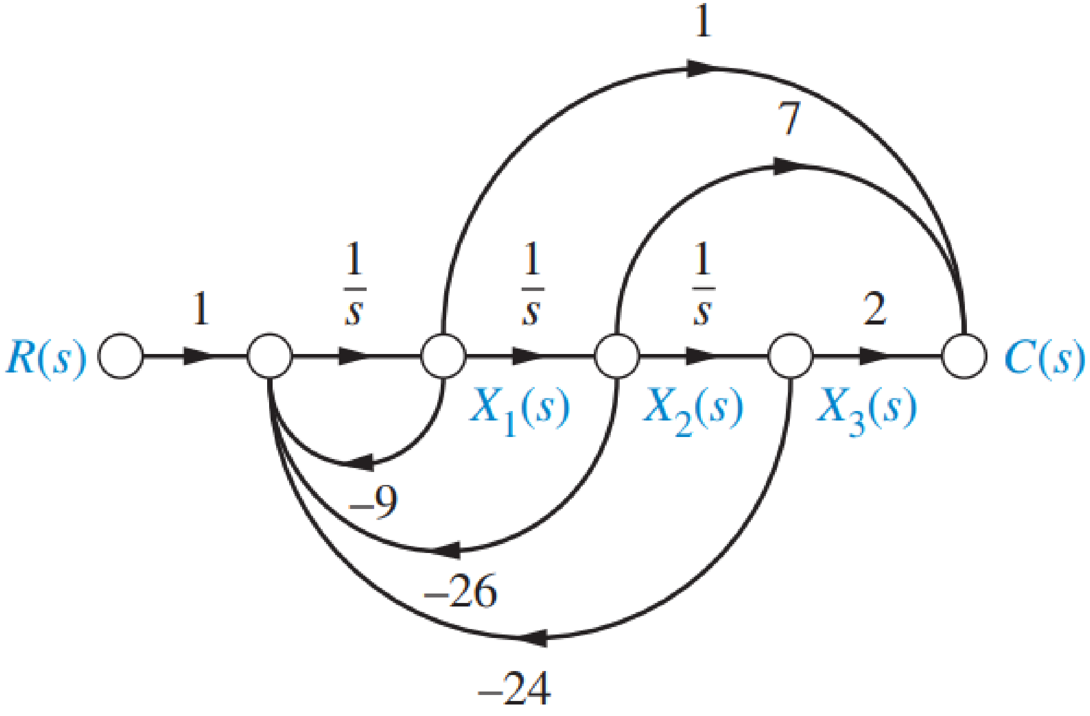
Phase Variable Form



$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -24 & -26 & -9\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 1\end{bmatrix} r$$

$$y = \begin{bmatrix}2 & 7 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

Controller Canonical Form



$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}-9 & -26 & -24\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}1\\ 0\\ 0\end{bmatrix} r$$

$$y = \begin{bmatrix}1 & 7 & 2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

Alternate Representation
[**4**] Observer Canonical Form

$$G(s) = \frac{C(s)}{R(s)} = \frac{s^2 + 7s + 2}{s^3 + 9s^2 + 26s + 24} = \frac{\frac{1}{s} + \frac{7}{s^2} + \frac{2}{s^3}}{1 + \frac{9}{s} + \frac{26}{s^2} + \frac{24}{s^3}}$$

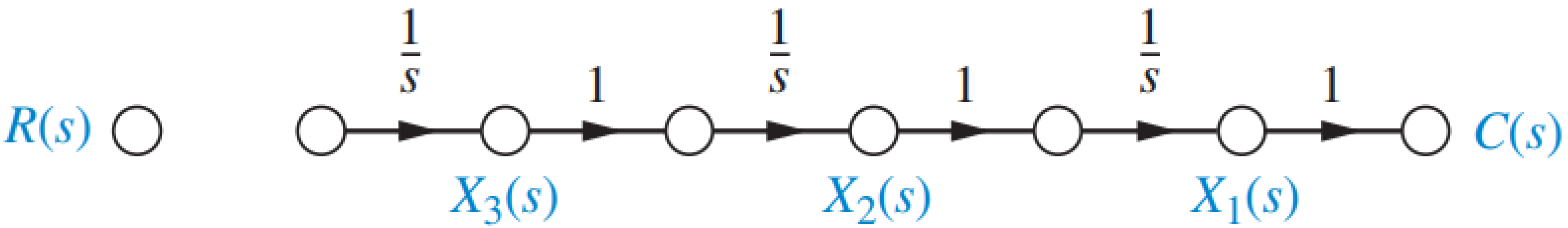
$$\left[\frac{1}{s} + \frac{7}{s^2} + \frac{2}{s^3}\right] R(s) = \left[1 + \frac{9}{s} + \frac{26}{s^2} + \frac{24}{s^3}\right] C(s)$$

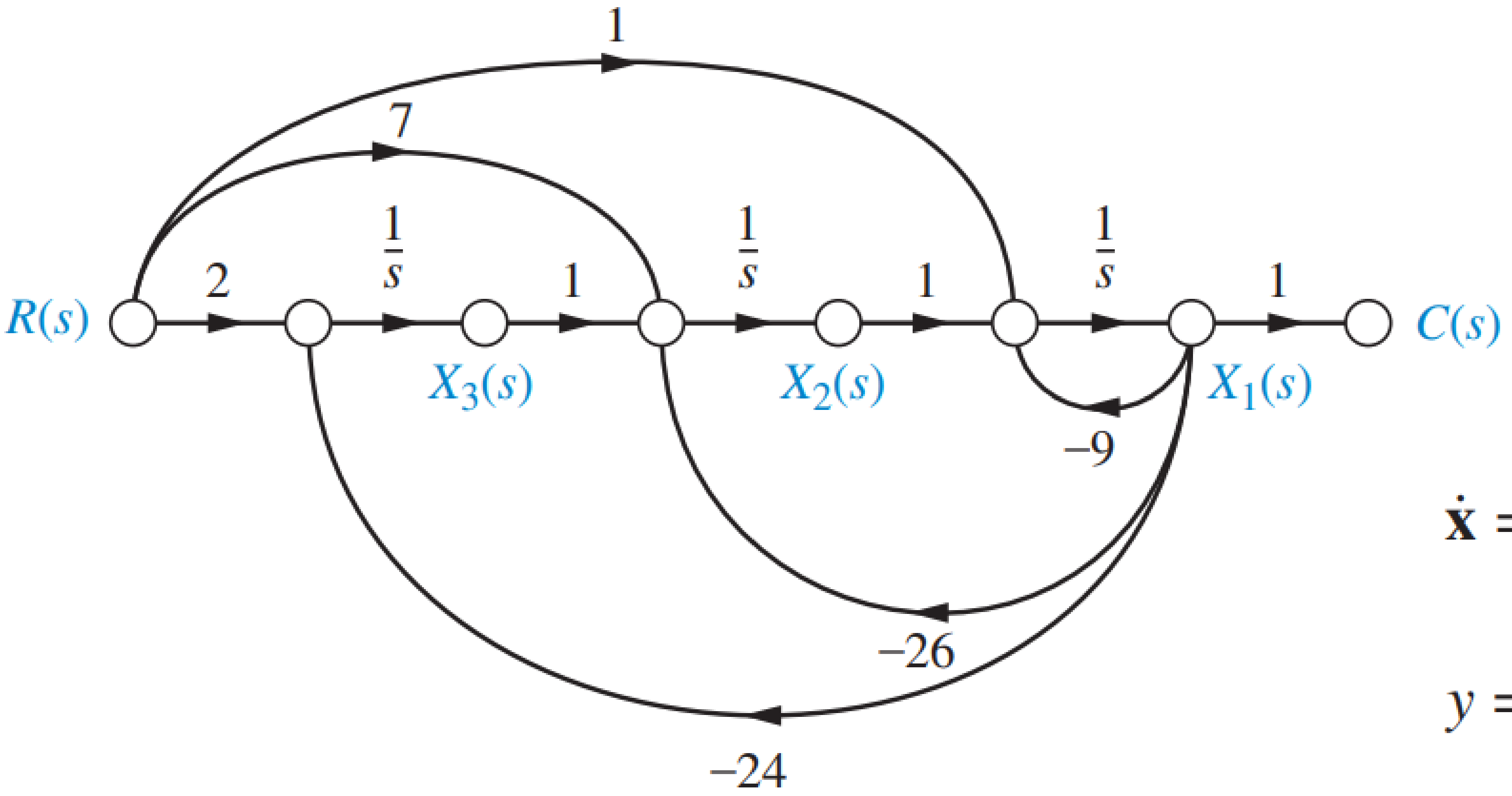
$$C(s) = \frac{1}{s}[R(s) - 9C(s)] + \frac{1}{s^2}[7R(s) - 26C(s)] + \frac{1}{s^3}[2R(s) - 24C(s)]$$

$$C(s) = \frac{1}{s}\left[[R(s) - 9C(s)] + \frac{1}{s}\left([7R(s) - 26C(s)] + \frac{1}{s}[2R(s) - 24C(s)]\right)\right]$$

continued…

$$C(s) = \frac{1}{s}\left[[R(s) - 9C(s)] + \frac{1}{s}\left([7R(s) - 26C(s)] + \frac{1}{s}[2R(s) - 24C(s)]\right)\right]$$





$$\dot{\mathbf{x}} = \begin{bmatrix} -9 & 1 & 0 \\ -26 & 0 & 1 \\ -24 & 0 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 \\ 7 \\ 2 \end{bmatrix} r$$

$$y = [\,1 \quad 0 \quad 0\,]\,\mathbf{x}$$

Stability

$$c(t) = c_{\text{forced}}(t) + c_{\text{natural}}(t)$$

<u>In terms of natural response:</u>

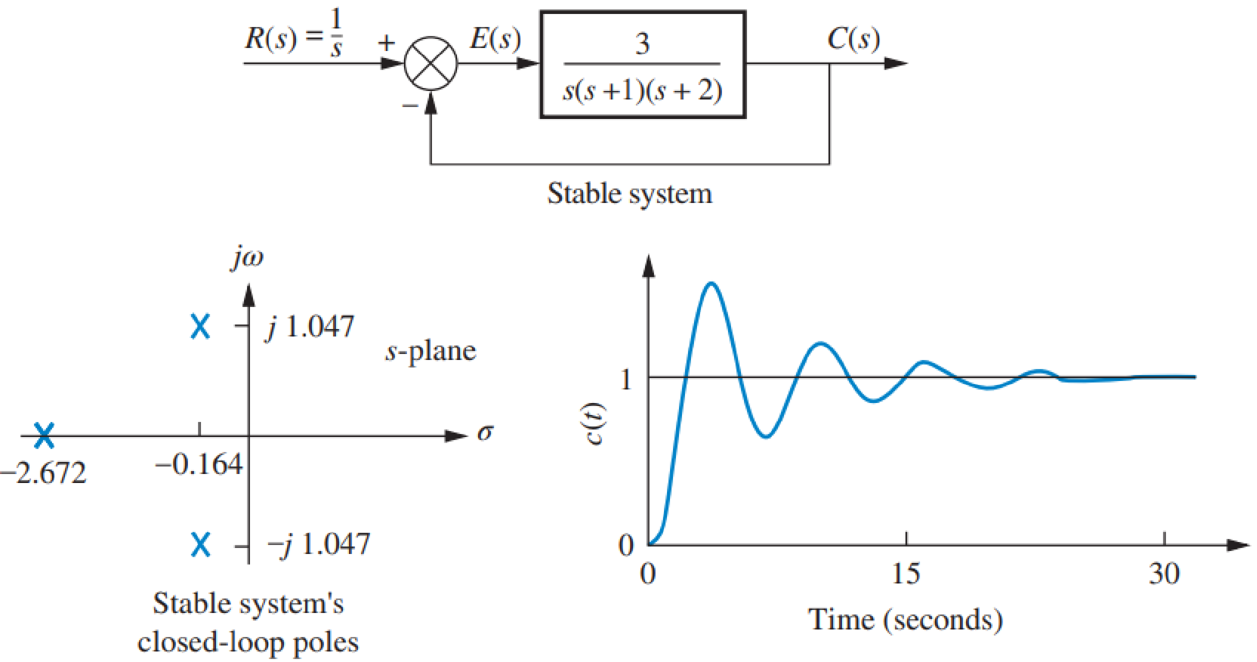
- ❖ A system is stable if the natural response approaches zero as time approaches infinity.
- ❖ A system is unstable if the natural response approaches infinity as time approaches infinity.
- ❖ A system is marginally stable if the natural response neither decays nor grows but remains constant or oscillates.

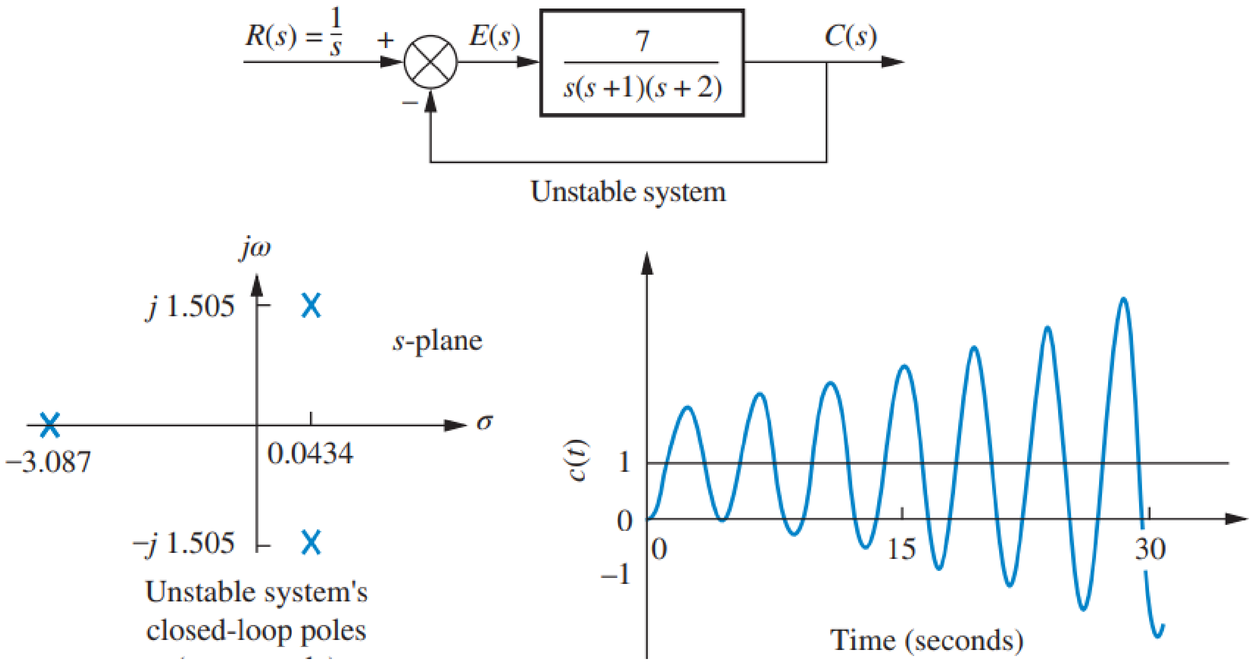
<u>In terms of total response:</u>

- ❖ A system is stable if *every* bounded input yields a bounded output.
- ❖ A system is unstable if *any* bounded input yields an unbounded output.

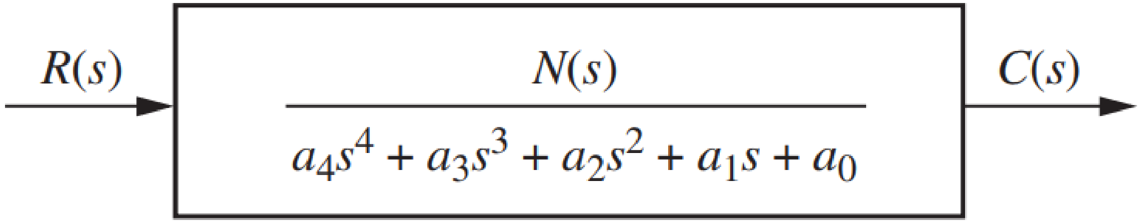
continued…

- Stable systems have CLTF with poles only in the LHP.
- Unstable systems have CLTF with at least one pole in the RHP *and/or* poles of multiplicity greater than 1 on the imaginary axis.
- Marginally stable systems have CLTF with only imaginary axis poles of multiplicity 1 and poles in the LHP.





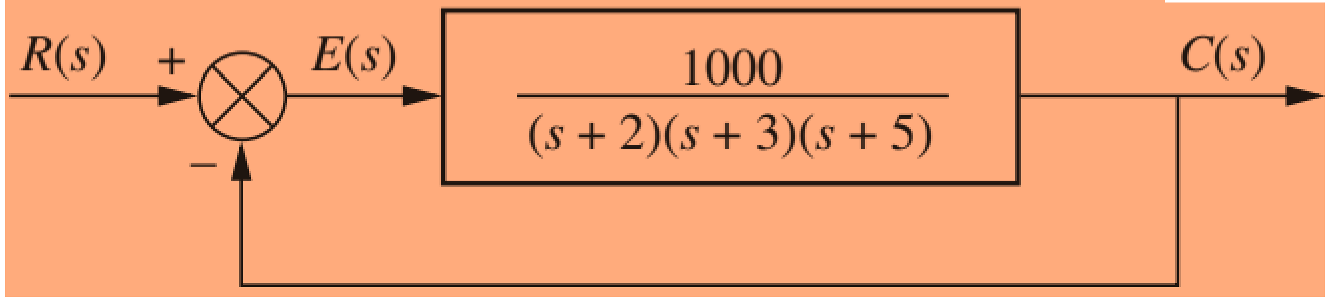
Routh-Hurwitz Criterion

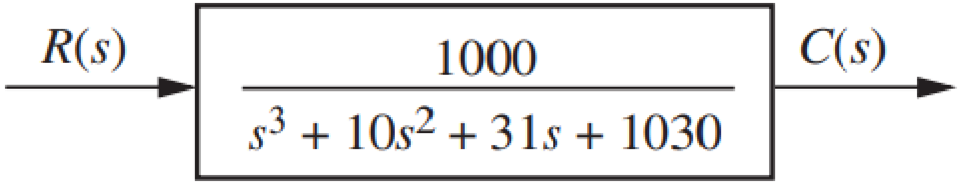


s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2	$\dfrac{-\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix}}{a_3} = b_1$	$\dfrac{-\begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix}}{a_3} = b_2$	$\dfrac{-\begin{vmatrix} a_4 & 0 \\ a_3 & 0 \end{vmatrix}}{a_3} = 0$
s^1	$\dfrac{-\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}}{b_1} = c_1$	$\dfrac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$	$\dfrac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$
s^0	$\dfrac{-\begin{vmatrix} b_1 & b_2 \\ c_1 & 0 \end{vmatrix}}{c_1} = d_1$	$\dfrac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$	$\dfrac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$

Example

Make the Routh table for the system shown





s^3	1	31	0
s^2	~~10~~ 1	~~1030~~ 103	0
s^1	$\dfrac{-\begin{vmatrix} 1 & 31 \\ 1 & 103 \end{vmatrix}}{1} = -72$	$\dfrac{-\begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix}}{1} = 0$	$\dfrac{-\begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix}}{1} = 0$
s^0	$\dfrac{-\begin{vmatrix} 1 & 103 \\ -72 & 0 \end{vmatrix}}{-72} = 103$	$\dfrac{-\begin{vmatrix} 1 & 0 \\ -72 & 0 \end{vmatrix}}{-72} = 0$	$\dfrac{-\begin{vmatrix} 1 & 0 \\ -72 & 0 \end{vmatrix}}{-72} = 0$

continued…

(# Roots in the RHP) = (# Sign changes in 1st column)

So, the previous system is unstable (two sign changes in 1st column).

Determine the stability of the closed-loop transfer function

$$T(s) = \frac{10}{s^5 + 2s^4 + 3s^3 + 6s^2 + 5s + 3}$$

				$\epsilon = +$
s^5	1	3	5	$+$
s^4	2	6	3	$+$
s^3	~~0~~ ϵ	$\frac{7}{2}$	0	$+$
s^2	$\frac{6\epsilon - 7}{\epsilon}$	3	0	$-$
s^1	$\frac{42\epsilon - 49 - 6\epsilon^2}{12\epsilon - 14}$	0	0	$+$
s^0	3	0	0	$+$

This system is unstable (two sign changes).

continued…

Determine the stability of the closed-loop transfer function

$$T(s) = \frac{10}{s^5 + 7s^4 + 6s^3 + 42s^2 + 8s + 56}$$

s^5	1	6	8
s^4	~~7~~ 1	~~42~~ 6	~~56~~ 8
s^3	~~0~~ ~~4~~ 1	~~0~~ ~~12~~ 3	~~0~~ ~~0~~ 0
s^2	3	8	0
s^1	$\frac{1}{3}$	0	0
s^0	8	0	0

$s^4 + 6s^2 + 8$

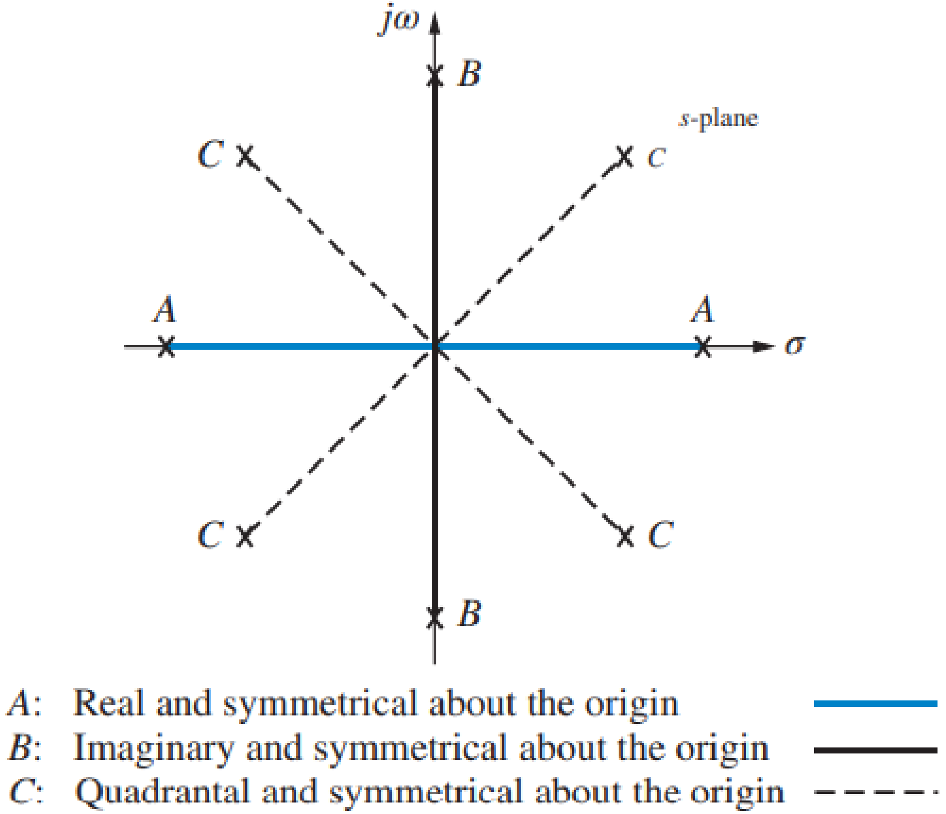
Differentiation

$4s^3 + 12s + 0$

This system is stable (no sign change).

continued…

- Even polynomial: has only even powers of *s* (e.g. $s^4 + 2s^2 + 2$).
- Even polynomial causes the row of zeros to appear.
  - The roots are symmetrical and real
  - The roots are symmetrical and imaginary
  - The roots are quadrantal
- If we do not have a row of zeros, we cannot possibly have jω roots.



continued...

tell how many poles are in the right half-plane, in the left half-plane, and on the $j\omega$-axis.

$$T(s) = \frac{20}{s^8 + s^7 + 12s^6 + 22s^5 + 39s^4 + 59s^3 + 48s^2 + 38s + 20}$$

s^8	1	12	39	48	20
s^7	1	22	59	38	0
s^6	~~−10~~ −1	~~−20~~ −2	~~10~~ 1	~~20~~ 2	0
s^5	~~20~~ 1	~~60~~ 3	~~40~~ 2	0	0
s^4	1	3	2	0	0
s^3	~~0~~ ~~4~~ 2	~~0~~ ~~6~~ 3	~~0~~ ~~0~~ 0	0	0
s^2	~~$\frac{3}{2}$~~ 3	~~2~~ 4	0	0	0
s^1	$\frac{1}{3}$	0	0	0	0
s^0	4	0	0	0	0

$s^4 + 3s^2 + 2$

$4s^3 + 6s + 0$

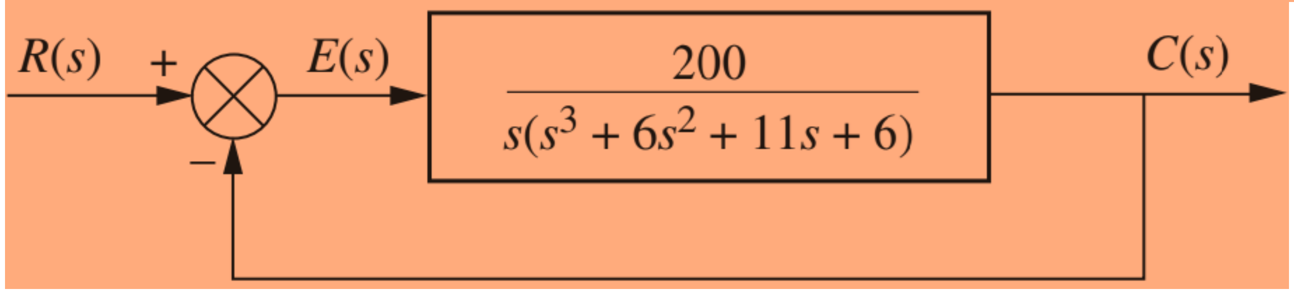
continued…

s^8	1	12	39	48	20
s^7	1	22	59	38	0
s^6	−~~10~~ −1	−~~20~~ −2	~~10~~ 1	~~20~~ 2	0
s^5	~~20~~ 1	~~60~~ 3	~~40~~ 2	0	0
s^4	1	3	2	0	0
s^3	~~0~~ ~~4~~ 2	~~0~~ ~~6~~ 3	~~0~~ ~~0~~ 0	0	0
s^2	~~$\frac{3}{2}$~~ 3	~~2~~ 4	0	0	0
s^1	$\frac{1}{3}$	0	0	0	0
s^0	4	0	0	0	0

	Polynomial		
Location	**Even (fourth-order)**	**Other (fourth-order)**	**Total (eighth-order)**
Right half-plane	0	2	2
Left half-plane	0	2	2
$j\omega$	4	0	4

continued…

tell how many poles are in the right half-plane, in the left half-plane, and on the $j\omega$-axis.



$$T(s) = \frac{200}{s^4 + 6s^3 + 11s^2 + 6s + 200}$$

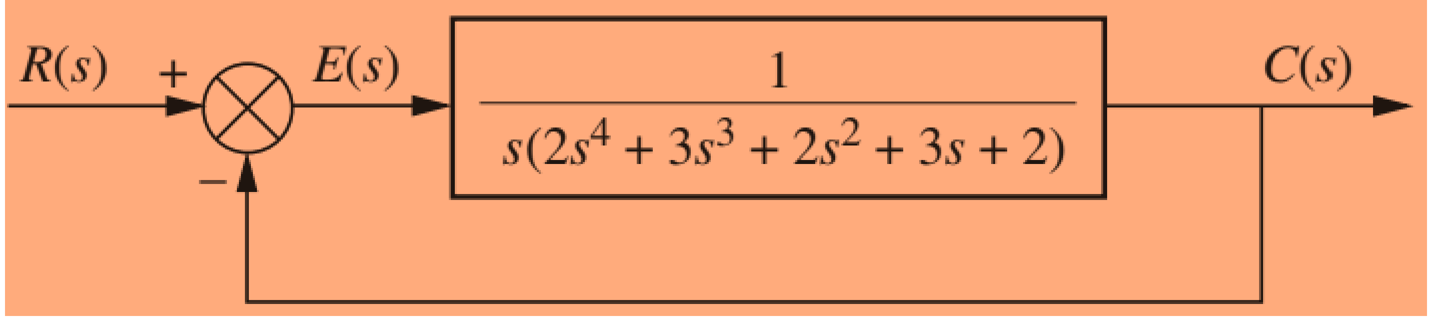
s^4	1	11 200
s^3	~~6~~ 1	~~6~~ 1
s^2	~~10~~ 1	~~200~~ 20
s^1	−19	
s^0	20	

- RHP poles: 2
- LHP poles: 2
- jω poles: 0

This system is unstable.

continued…

tell how many poles are in the right half-plane, in the left half-plane, and on the $j\omega$-axis.



$$T(s) = \frac{1}{2s^5 + 3s^4 + 2s^3 + 3s^2 + 2s + 1}$$

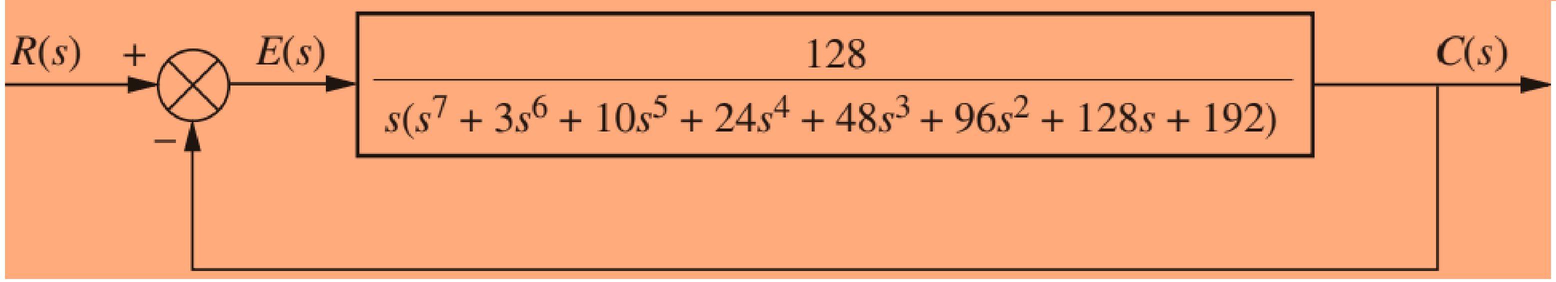
s^5	2	2	2	+
s^4	3	3	1	+
s^3	~~0~~ ϵ	$\frac{4}{3}$		+
s^2	$\frac{3\epsilon - 4}{\epsilon}$	1		−
s^1	$\frac{12\epsilon - 16 - 3\epsilon^2}{9\epsilon - 12}$			+
s^0	1			+

- RHP poles: 2
- LHP poles: 3
- jω poles: 0

This system is unstable.

continued…

tell how many poles are in the right half-plane, in the left half-plane, and on the $j\omega$-axis.



$$T(s) = \frac{128}{s^8 + 3s^7 + 10s^6 + 24s^5 + 48s^4 + 96s^3 + 128s^2 + 192s + 128}$$

s^8	1	10	48	128	128
s^7	~~3~~ 1	~~24~~ 8	~~96~~ 32	~~192~~ 64	
s^6	~~2~~ 1	~~16~~ 8	~~64~~ 32	~~128~~ 64	
s^5	~~0~~ ~~6~~ 3	~~0~~ ~~32~~ 16	~~0~~ ~~64~~ 32	~~0~~ ~~0~~ 0	
s^4	~~$\frac{8}{3}$~~ 1	~~$\frac{64}{3}$~~ 8	~~64~~ 24		
s^3	~~−8~~ −1	~~−40~~ −5			
s^2	~~3~~ 1	~~24~~ 8			
s^1	3				
s^0	8				

$s^6 + 8s^4 + 32s^2 + 64$

$6s^5 + 32s^3 + 64s + 0$

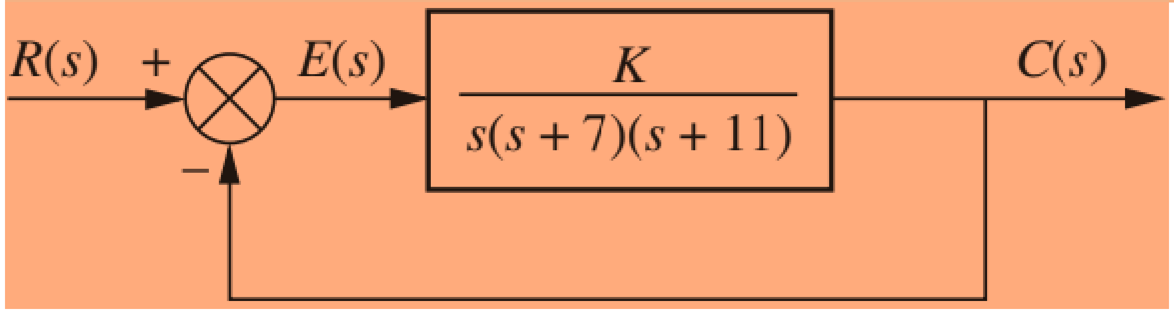
continued...

s^8	1	10	48	128	128
s^7	~~3~~ 1	~~24~~ 8	~~96~~ 32	~~192~~ 64	
s^6	~~2~~ 1	~~16~~ 8	~~64~~ 32	~~128~~ 64	
s^5	~~0~~ ~~6~~ 3	~~0~~ ~~32~~ 16	~~0~~ ~~64~~ 32	~~0~~ ~~0~~ 0	
s^4	~~$\frac{8}{3}$~~ 1	~~$\frac{64}{3}$~~ 8	~~64~~ 24		
s^3	~~−8~~ −1	~~−40~~ −5			
s^2	~~3~~ 1	~~24~~ 8			
s^1	3				
s^0	8				

	Polynomial		
Location	**Even (sixth-order)**	**Other (second-order)**	**Total (eighth-order)**
Right half-plane	2	0	2
Left half-plane	2	2	4
$j\omega$	2	0	2

continued…

Find the range of gain, K, that will cause the system to be stable and unstable. Assume K > 0.

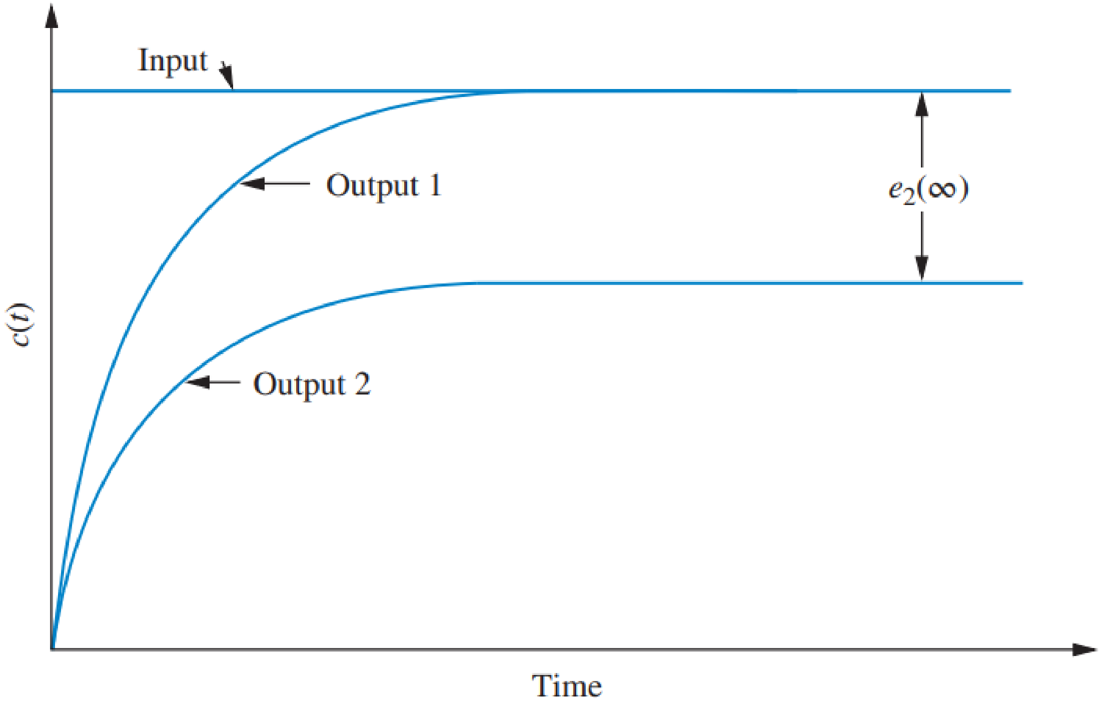


$$T(s) = \frac{K}{s^3 + 18s^2 + 77s + K}$$

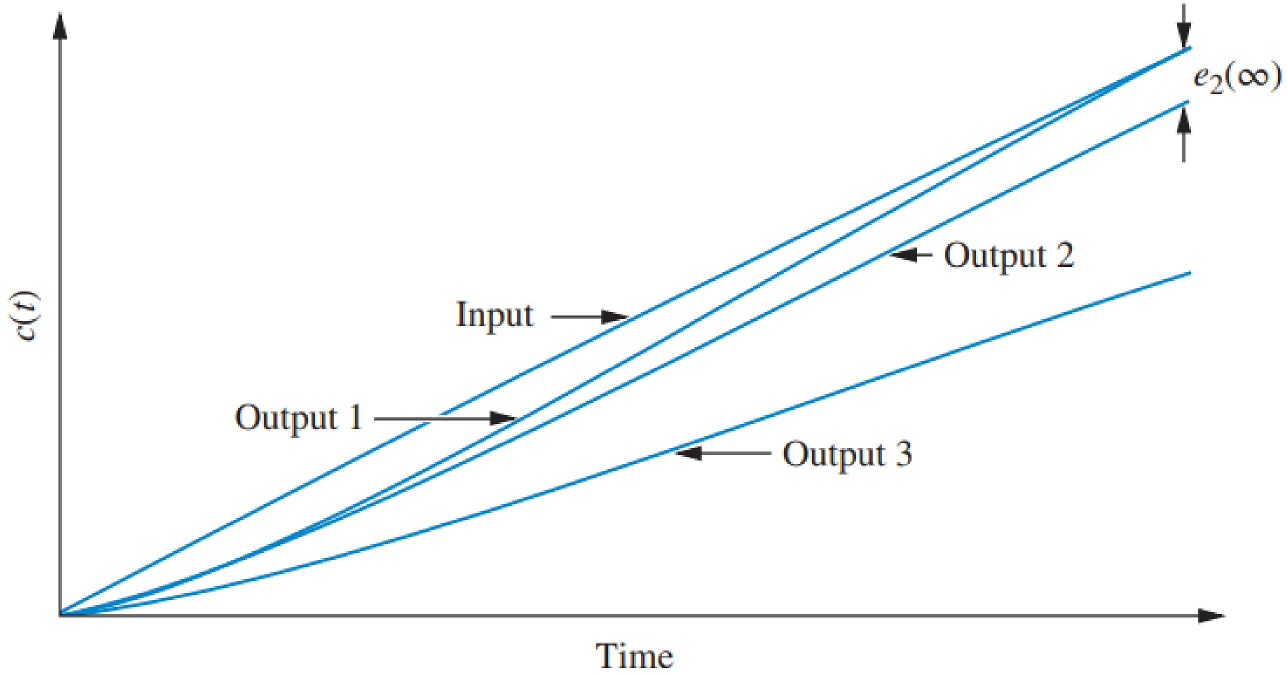
s^3	1	77
s^2	18	K
s^1	$\frac{1386 - K}{18}$	
s^0	K	

- If K < 1386, all terms in the 1st column is positive. The system is **stable**.
- If K > 1386, the s^1 term in the 1st column is negative. The system is **unstable**.

Steady-State Error



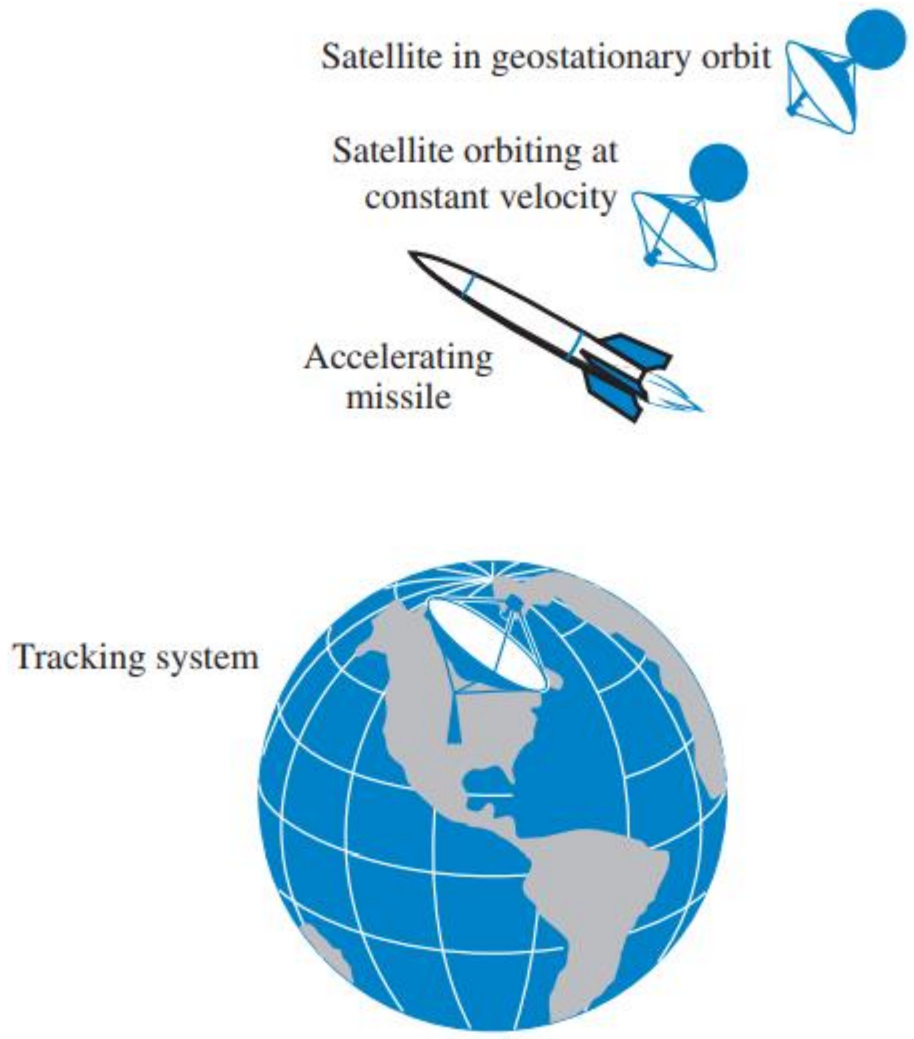
Step Input Response

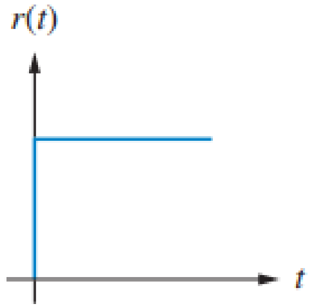
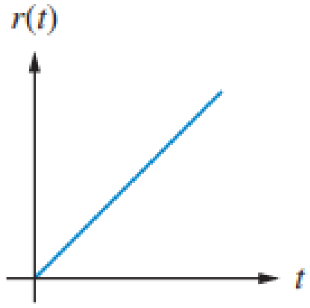
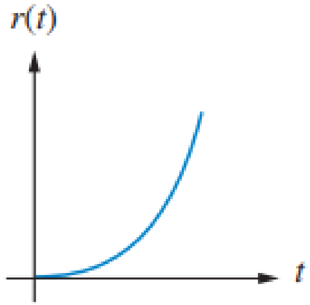


Ramp Input Response

Steady-state error (SSE) is the difference between the input and the output for a prescribed test input as $t \to \infty$.

Test Waveform Example



Waveform	Name	Physical interpretation
	Step	Constant position
	Ramp	Constant velocity
	Parabola	Constant acceleration

SSE for Unity Negative Feedback System

- <u>In terms of T(s)</u>

$$E(s) = R(s) - C(s)$$

$$C(s) = R(s)T(s)$$

$$E(s) = R(s)[1 - T(s)]$$

Applying the final value theorem,

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$$

$$e(\infty) = \lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s) = \lim_{s \to 0} sR(s)[1 - T(s)]$$

- <u>In terms of G(s)</u>

$$C(s) = E(s)G(s)$$

$$E(s) = \frac{R(s)}{1 + G(s)}$$

$$e(\infty) = \lim_{s \to 0} \frac{sR(s)}{1 + G(s)}$$

continued...

$$e(\infty) = \lim_{s\to 0}\frac{sR(s)}{1+G(s)}$$

- <u>Step Input</u>

$R(s) = 1/s$

$$e(\infty) = e_{\text{step}}(\infty) = \lim_{s\to 0}\frac{s(1/s)}{1+G(s)} = \frac{1}{1+\lim_{s\to 0} G(s)}$$

- <u>Ramp Input</u>

$R(s) = 1/s^2$

$$e(\infty) = e_{\text{ramp}}(\infty) = \lim_{s\to 0}\frac{s\left(1/s^2\right)}{1+G(s)} = \lim_{s\to 0}\frac{1}{s+sG(s)} = \frac{1}{\lim_{s\to 0} sG(s)}$$

- <u>Parabolic Input</u>

$R(s) = 1/s^3$

$$e(\infty) = e_{\text{parabola}}(\infty) = \lim_{s\to 0}\frac{s\left(1/s^3\right)}{1+G(s)} = \lim_{s\to 0}\frac{1}{s^2+s^2G(s)} = \frac{1}{\lim_{s\to 0} s^2G(s)}$$

Example

Find the steady-state error for the system $T(s) = 5/(s^2 + 7s + 10)$ the input is a unit step.

$$R(s) = 1/s$$

$$E(s) = R(s)[1 - T(s)]$$

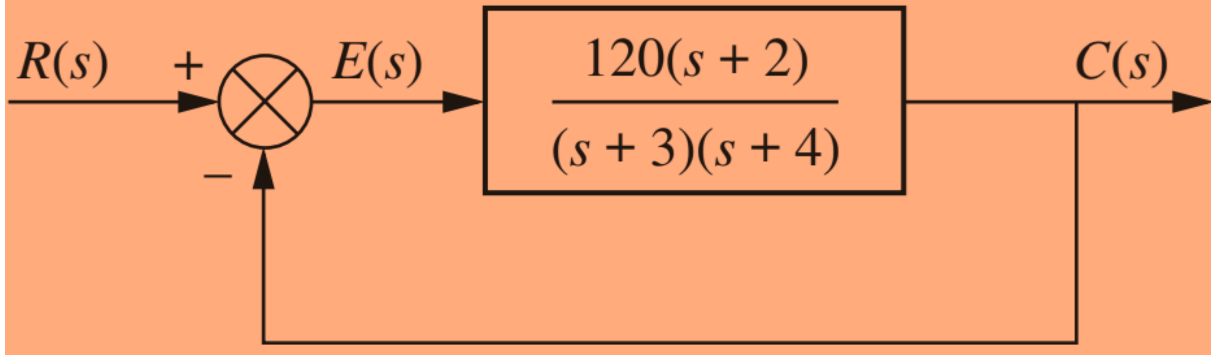
$$= \frac{s^2 + 7s + 5}{s(s^2 + 7s + 10)}$$

$$e(\infty) = \lim_{s \to 0} sE(s)$$

$$= 1/2.$$

continued…

Find the steady-state errors for inputs of $5u(t)$, $5tu(t)$, and $5t^2u(t)$ to the system shown



$5u(t) \longleftrightarrow 5/s$

$5tu(t) \longleftrightarrow 5/s^2$

$5t^2u(t) \longleftrightarrow 10/s^3$

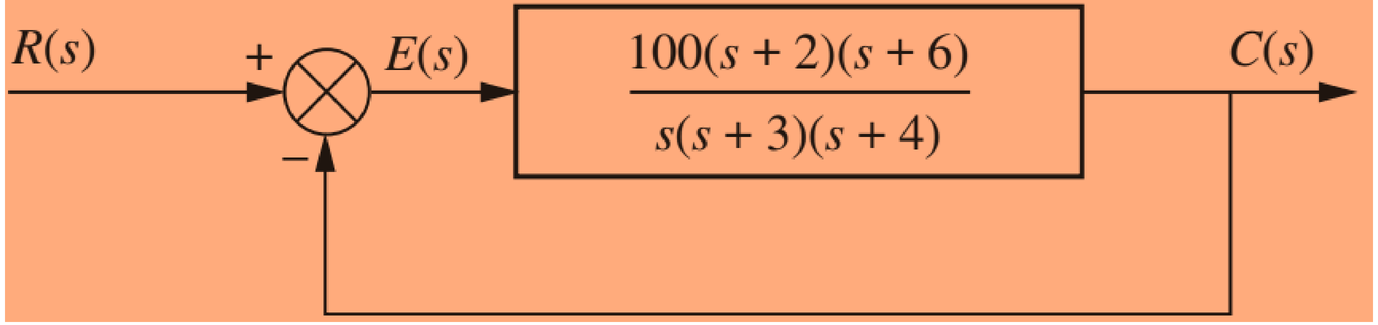
$$e_{\text{step}}(\infty) = \frac{5}{1 + \lim_{s\to 0} G(s)} = \frac{5}{1+20} = \frac{5}{21}$$

$$e_{\text{ramp}}(\infty) = \frac{5}{\lim_{s\to 0} sG(s)} = \frac{5}{0} = \infty$$

$$e_{\text{parabola}}(\infty) = \frac{10}{\lim_{s\to 0} s^2G(s)} = \frac{10}{0} = \infty$$

continued…

Find the steady-state errors for inputs of $5u(t)$, $5tu(t)$, and $5t^2u(t)$ to the system shown



$$e_{\text{step}}(\infty) = \frac{5}{1 + \lim_{s\to 0} G(s)} = \frac{5}{\infty} = 0$$

$$e_{\text{ramp}}(\infty) = \frac{5}{\lim_{s\to 0} sG(s)} = \frac{5}{100} = \frac{1}{20}$$

$$e_{\text{parabola}}(\infty) = \frac{10}{\lim_{s\to 0} s^2G(s)} = \frac{10}{0} = \infty$$

Static Error Constant

$$e_{\text{step}}(\infty) = \frac{1}{1 + \lim_{s \to 0} G(s)} \rightarrow K_p = \lim_{s \to 0} G(s)$$

position constant,

$$e_{\text{ramp}}(\infty) = \frac{1}{\lim_{s \to 0} sG(s)} \rightarrow K_v = \lim_{s \to 0} sG(s)$$

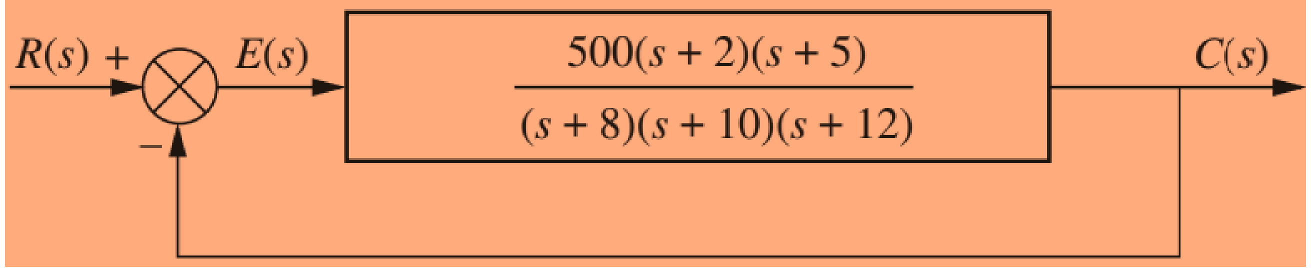
velocity constant,

$$e_{\text{parabola}}(\infty) = \frac{1}{\lim_{s \to 0} s^2 G(s)} \rightarrow K_a = \lim_{s \to 0} s^2 G(s)$$

acceleration constant,

Example

evaluate the static error constants and find the expected error for the standard step, ramp, and parabolic inputs.



$$K_p = \lim_{s\to 0} G(s) = \frac{500 \times 2 \times 5}{8 \times 10 \times 12} = 5.208$$

$$K_v = \lim_{s\to 0} sG(s) = 0$$

$$K_a = \lim_{s\to 0} s^2 G(s) = 0$$

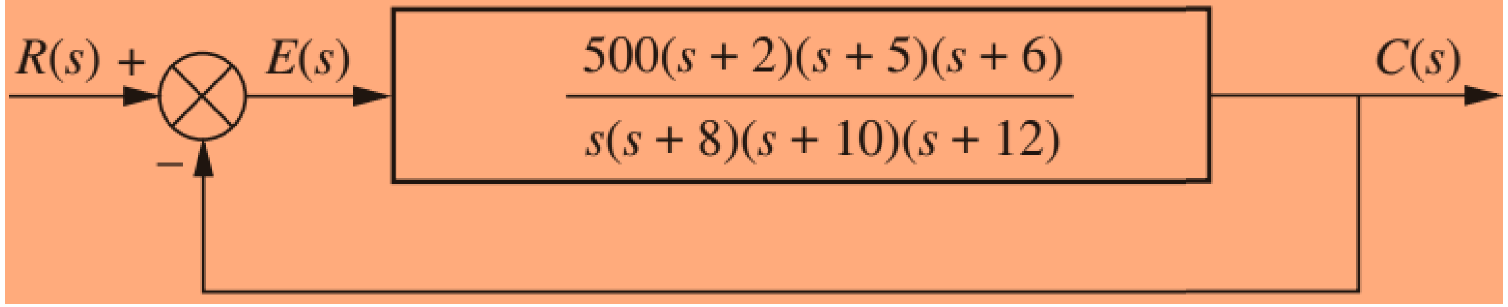
$$e_{step}(\infty) = \frac{1}{1 + K_p} = 0.161$$

$$e_{ramp}(\infty) = \frac{1}{K_v} = \infty$$

$$e_{parabola}(\infty) = \frac{1}{K_a} = \infty$$

continued...

evaluate the static error constants and find the expected error for the standard step, ramp, and parabolic inputs.



$$K_p = \lim_{s\to 0} G(s) = \infty$$

$$K_v = \lim_{s\to 0} sG(s) = \frac{500\times 2\times 5\times 6}{8\times 10\times 12} = 31.25$$

$$K_a = \lim_{s\to 0} s^2 G(s) = 0$$

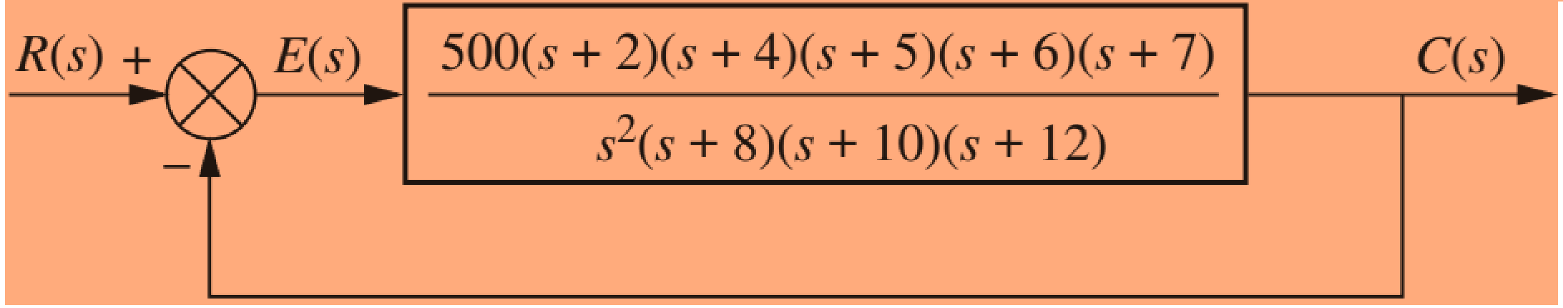
$$e_{\text{step}}(\infty) = \frac{1}{1+K_p} = 0$$

$$e_{\text{ramp}}(\infty) = \frac{1}{K_v} = 0.032$$

$$e_{\text{parabola}}(\infty) = \frac{1}{K_a} = \infty$$

continued…

evaluate the static error constants and find the expected error for the standard step, ramp, and parabolic inputs.



$$K_p = \lim_{s\to 0} G(s) = \infty$$

$$K_v = \lim_{s\to 0} sG(s) = \infty$$

$$K_a = \lim_{s\to 0} s^2 G(s) = \frac{500\times 2\times 4\times 5\times 6\times 7}{8\times 10\times 12} = 875$$

$$e_{\text{step}}(\infty) = \frac{1}{1+K_p} = 0$$

$$e_{\text{ramp}}(\infty) = \frac{1}{K_v} = 0$$

$$e_{\text{parabola}}(\infty) = \frac{1}{K_a} = 1.14\times 10^{-3}$$

System Type

- Number of pure integrations in the forward path

Type 0		Type 1		Type 2	
Static error constant	**Error**	**Static error constant**	**Error**	**Static error constant**	**Error**
$K_p = \text{Constant}$	$\frac{1}{1+K_p}$	$K_p = \infty$	0	$K_p = \infty$	0
$K_v = 0$	∞	$K_v = \text{Constant}$	$\frac{1}{K_v}$	$K_v = \infty$	0
$K_a = 0$	∞	$K_a = 0$	∞	$K_a = \text{Constant}$	$\frac{1}{K_a}$

Example

What information is contained in the specification $K_v = 1000$

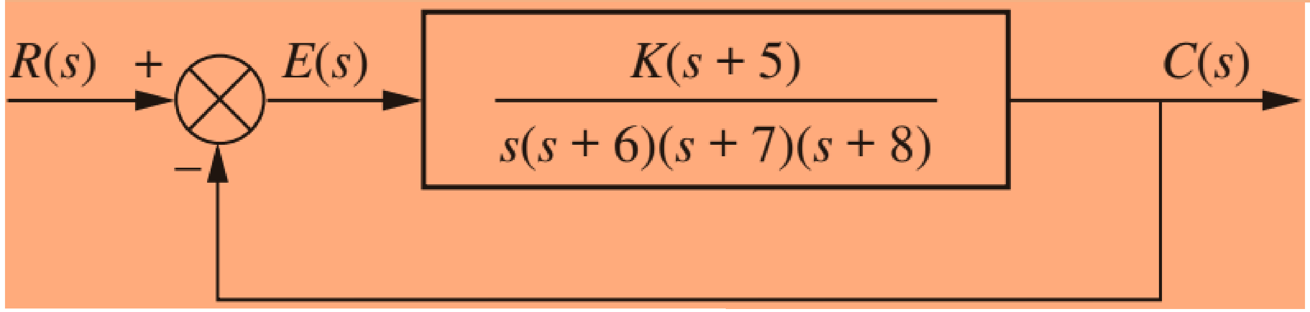
- The system is stable.
- The system is of Type 1.
- A ramp input is the test signal.
- The SSE is $1/K_v$ per unit ramp.

What information is contained in the specification $K_p = 1000$?

- The system is stable.
- The system is of Type 0.
- A step input is the test signal.
- The SSE is $1/(1 + K_p)$ per unit step.

continued…

Find the value of K so that there is 10% error in the steady state.

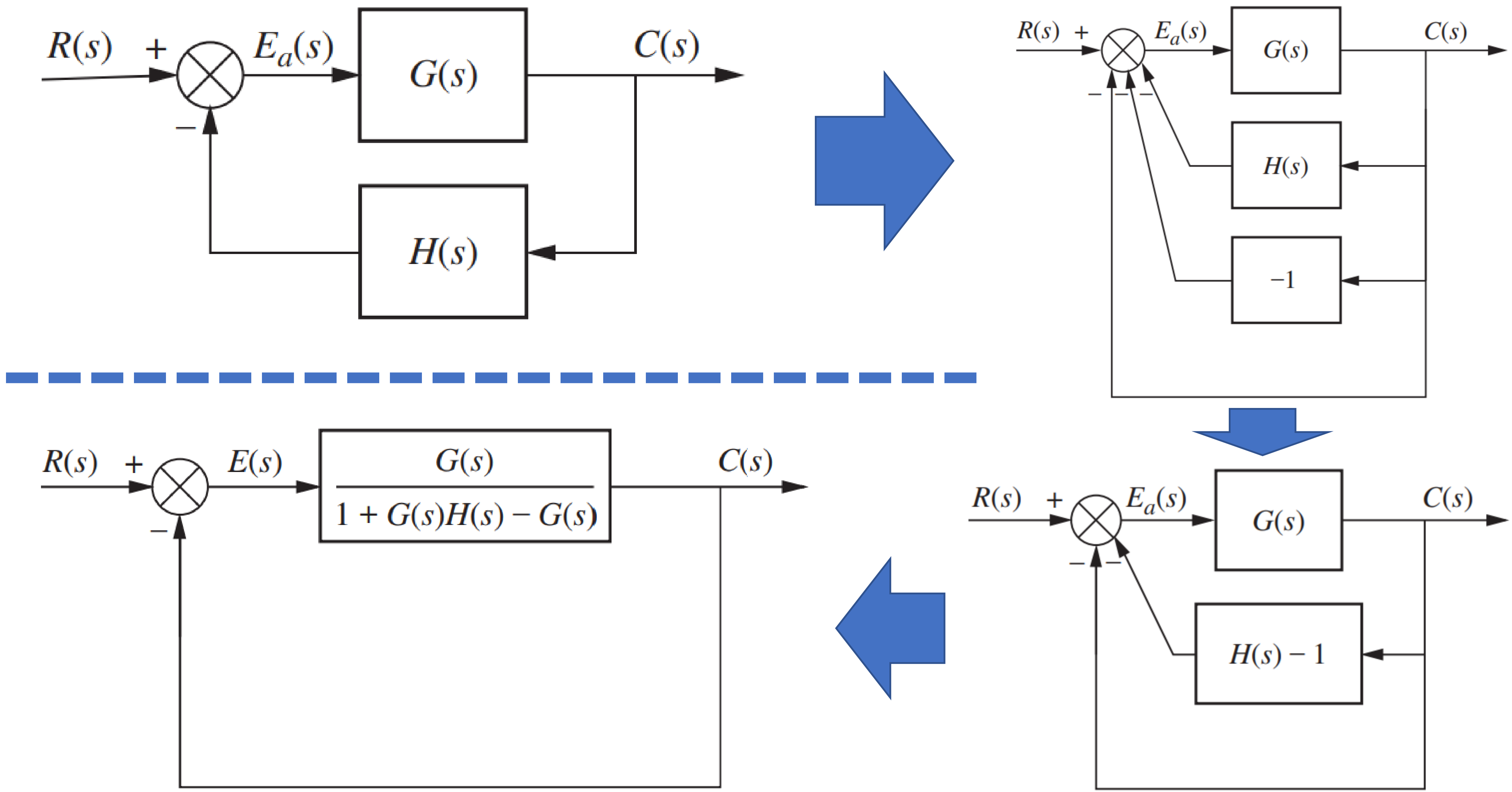


$$e(\infty) = \frac{1}{K_v} = 0.1 \longrightarrow K_v = 10 = \lim_{s \to 0} sG(s)$$

$$10 = \frac{K \times 5}{6 \times 7 \times 8}$$

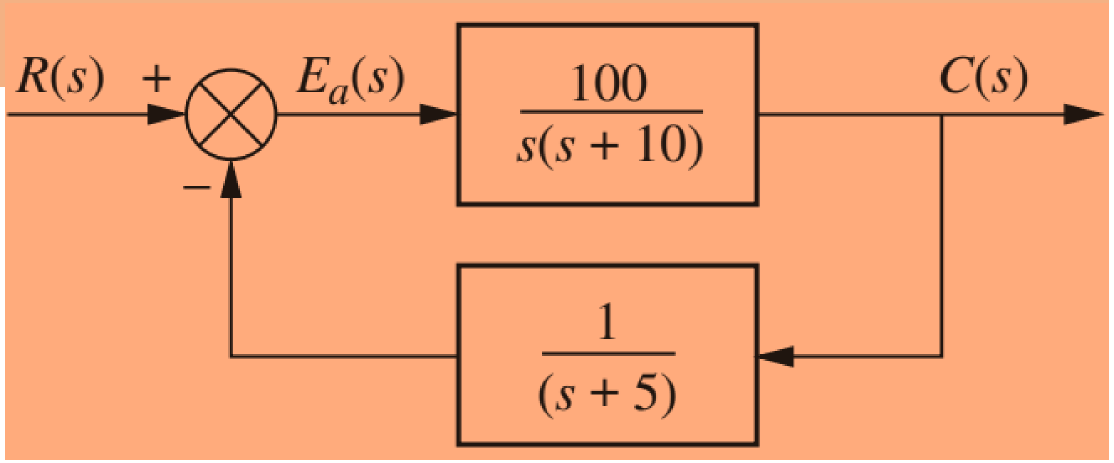
$$K = 672$$

SSE for Non-unity Negative Feedback System



Example

For the system shown, find the system type, the appropriate error constant associated with the system type, and the steady-state error for a unit step input.



$$G_e(s) = \frac{G(s)}{1 + G(s)H(s) - G(s)}$$

$$= \frac{100(s+5)}{s^3 + 15s^2 - 50s - 400}$$

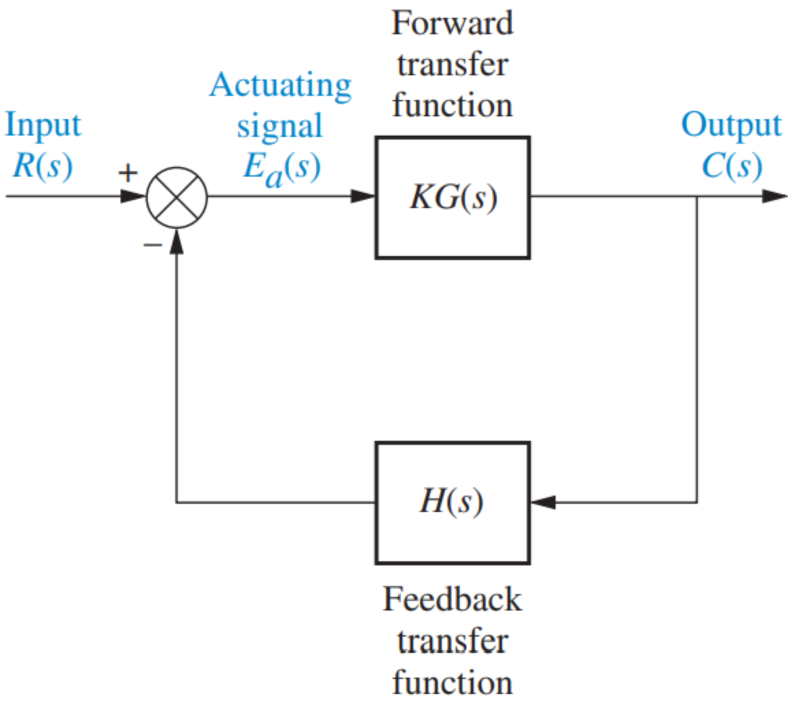
the system is Type 0

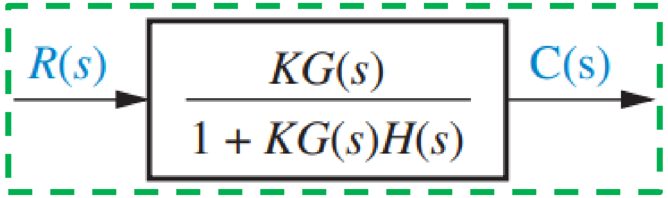
$$K_p = \lim_{s \to 0} G_e(s) = \frac{100 \times 5}{-400} = -\frac{5}{4}$$

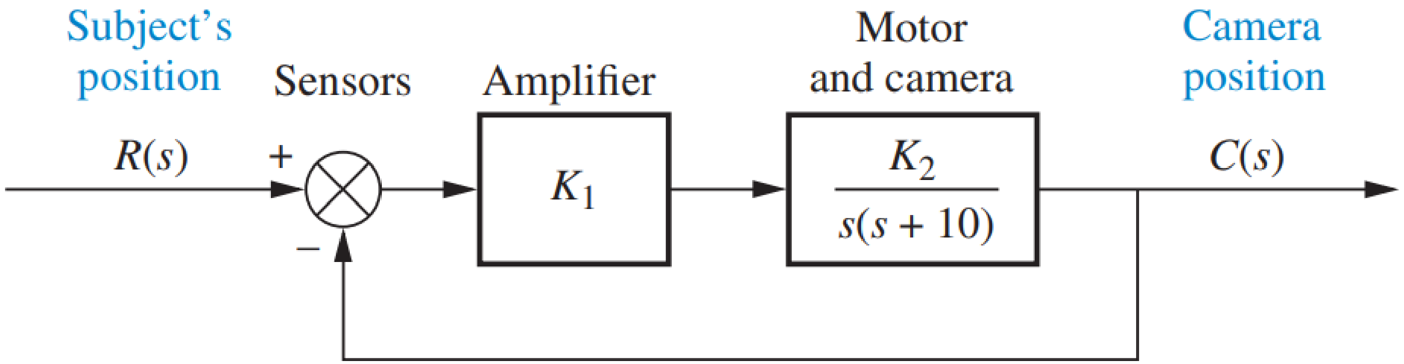
$$e(\infty) = \frac{1}{1 + K_p} = \frac{1}{1 - (5/4)} = -4$$

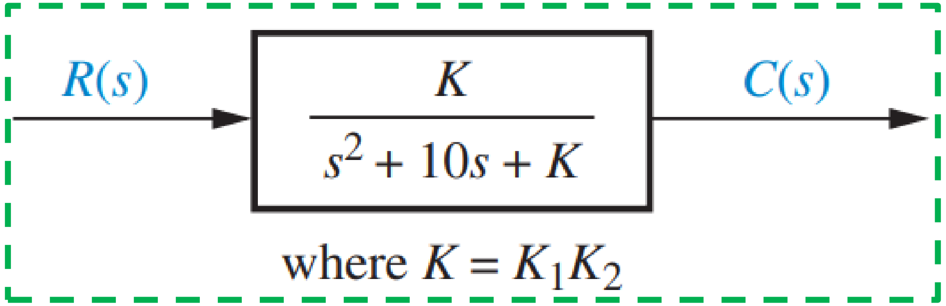
The negative value for SSE implies that the output step is *larger* than the input step.

Root Locus

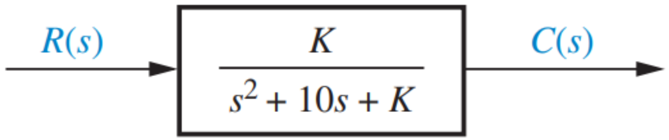

Input
R(s)
+
−
Actuating signal
Ea(s)
Forward transfer function
KG(s)
Output
C(s)
H(s)
Feedback transfer function

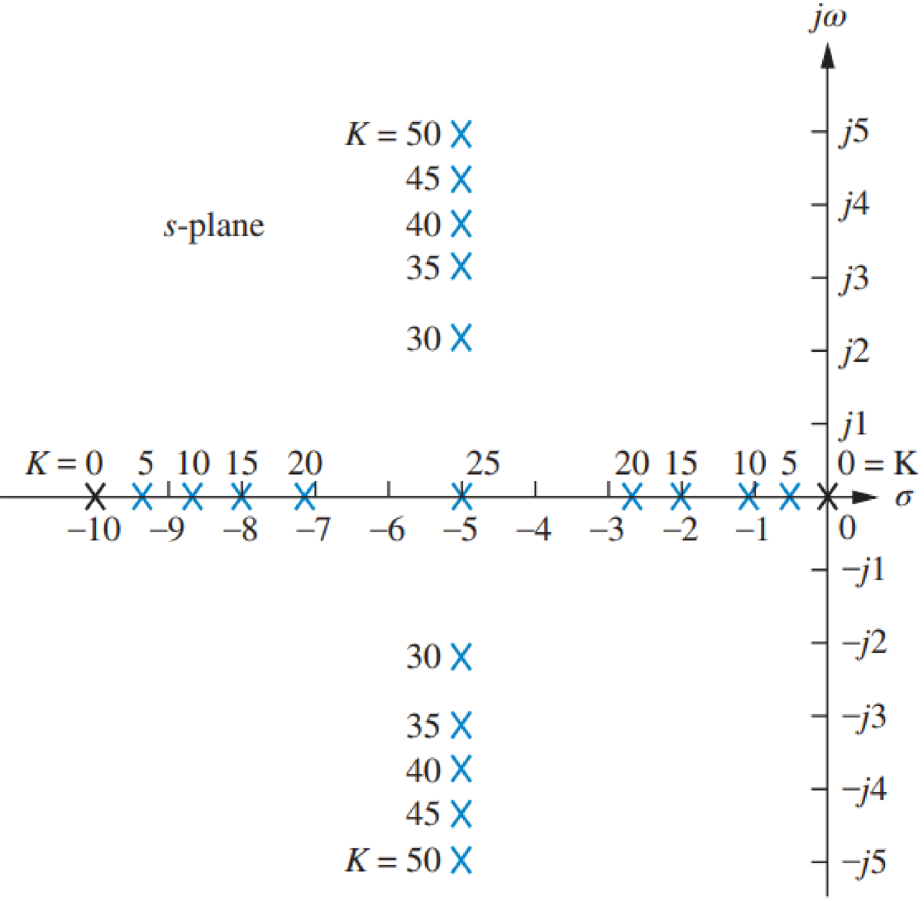

R(s)
KG(s) / (1 + KG(s)H(s))
C(s)


Subject's position
R(s)
Sensors
+
−
Amplifier
K1
Motor and camera
K2 / s(s + 10)
Camera position
C(s)

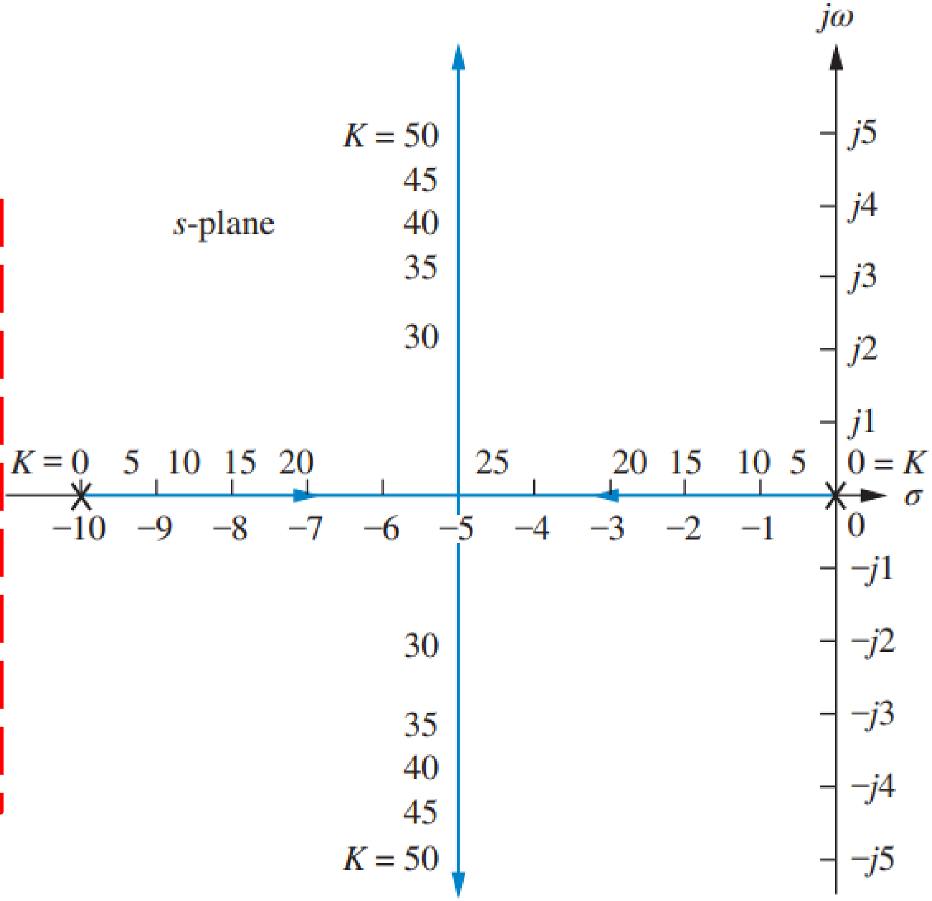

R(s)
K / (s2 + 10s + K)
C(s)
where K = K1K2

continued...



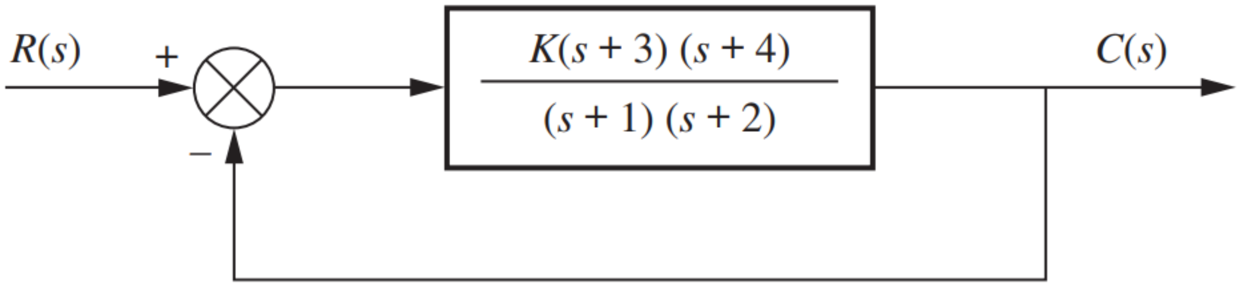


K	Pole 1	Pole 2
0	−10	0
5	−9.47	−0.53
10	−8.87	−1.13
15	−8.16	−1.84
20	−7.24	−2.76
25	−5	−5
30	$-5 + j2.24$	$-5 - j2.24$
35	$-5 + j3.16$	$-5 - j3.16$
40	$-5 + j3.87$	$-5 - j3.87$
45	$-5 + j4.47$	$-5 - j4.47$
50	$-5 + j5$	$-5 - j5$

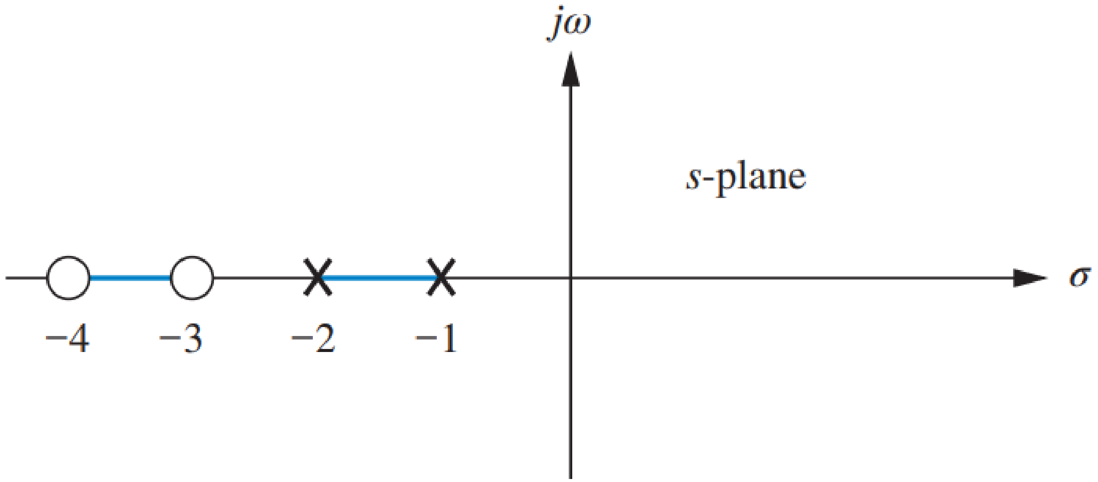


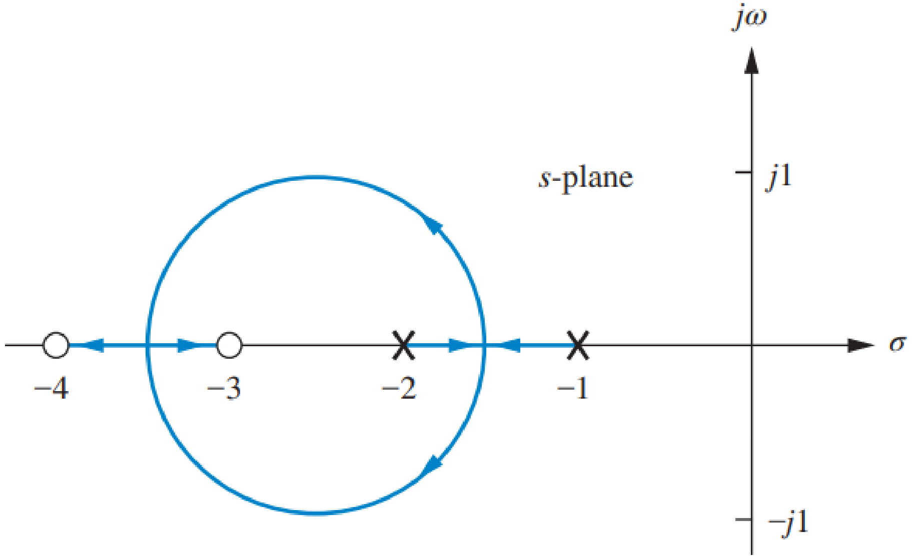
It is this representation of the paths of the closed-loop poles (as the *gain* is varied) that we call a **root locus**.

Sketching the Root Locus



- The number of branches equals the number of closed-loop poles.
- The root locus is symmetrical about the real axis.
- On the real axis, for $K > 0$, the root locus exists to the left of an odd number of real-axis, finite open-loop poles and/or finite open-loop zeros.
- The root locus begins at the finite and infinite poles of $G(s)H(s)$ and ends at the finite and infinite zeros of $G(s)H(s)$.





Root Locus: Behavior at Infinity

A function can also have poles and/or zeros at ∞.

- If the function approaches ∞ as s approaches ∞, then the function has a pole at ∞.
- If the function approaches 0 as s approaches ∞, then the function has a zero at ∞.

Example: $G(s) = s$ has a pole at ∞ ; $G(s) = 1/s$ has a zero at ∞.

➢ The root locus approaches straight lines (as asymptotes) as the locus approaches ∞.

$$\sigma_a = \frac{\sum \text{finite poles} - \sum \text{finite zeros}}{\#\,\text{finite poles} - \#\,\text{finite zeros}}$$

real-axis intercept,

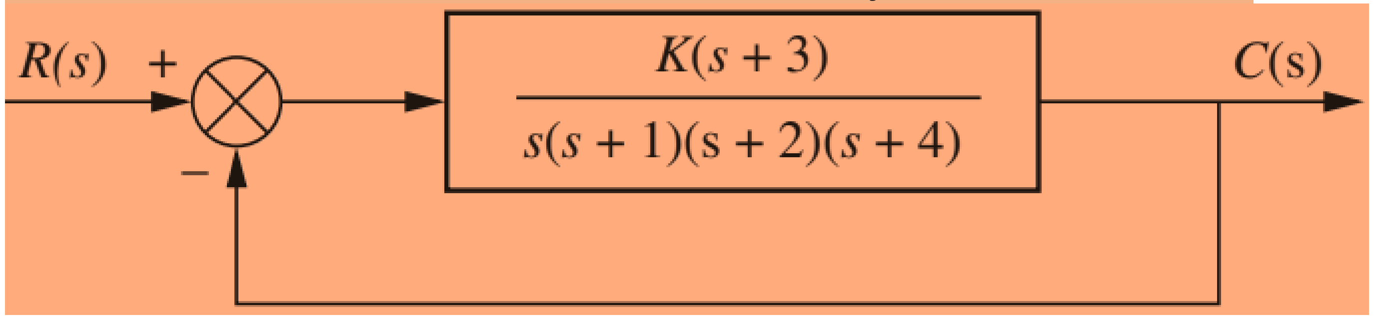
$$\theta_a = \frac{(2k+1)\pi}{\#\,\text{finite poles} - \#\,\text{finite zeros}}$$

angle,

$$k = 0, \pm 1, \pm 2, \pm 3$$

Example

Sketch the root locus for the system shown



$$\sigma_a = \frac{\sum \text{finite poles} - \sum \text{finite zeros}}{\#\,\text{finite poles} - \#\,\text{finite zeros}}$$

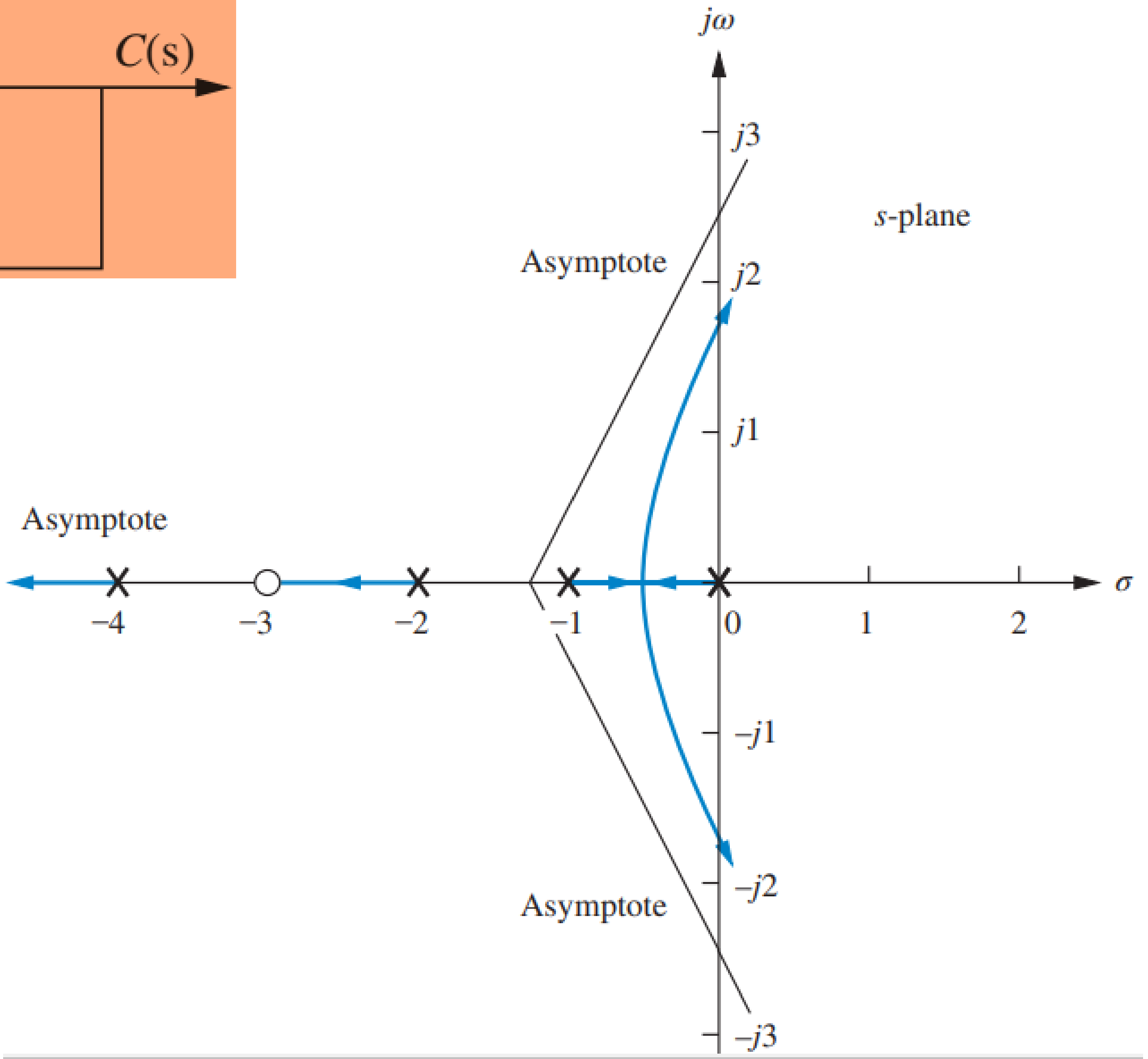
$$= \frac{(-1-2-4)-(-3)}{4-1} = -\frac{4}{3}$$

$$\theta_a = \frac{(2k+1)\pi}{\#\,\text{finite poles} - \#\,\text{finite zeros}}$$

$$= \pi/3 \qquad \text{for } k = 0$$

$$= \pi \qquad \text{for } k = 1$$

$$= 5\pi/3 \qquad \text{for } k = 2$$



continued…

Sketch the root locus and its asymptotes for a unity negative feedback system that has the forward transfer function:

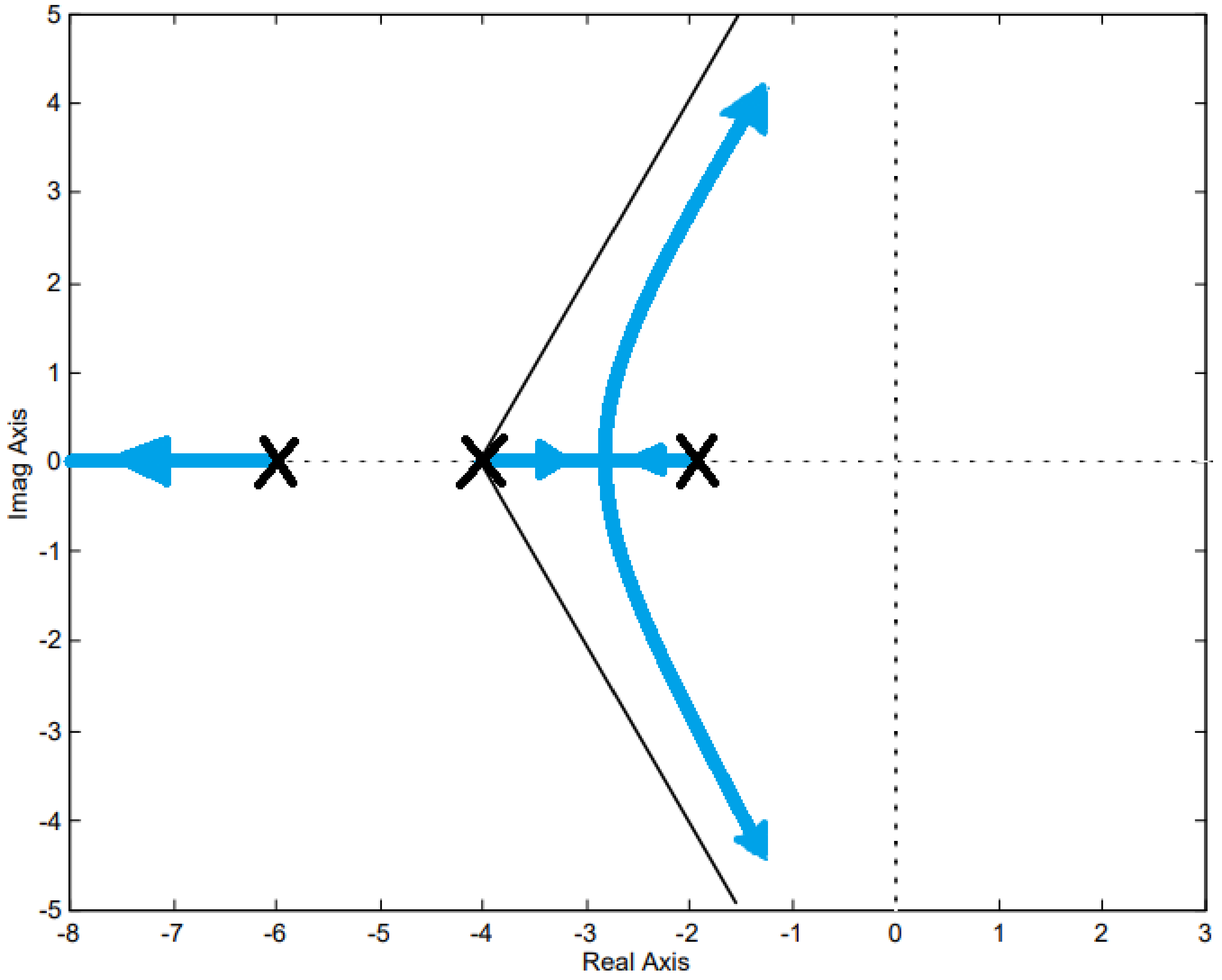
$$G(s) = \frac{K}{(s+2)(s+4)(s+6)}$$

$$\sigma_a = \frac{\sum \text{finite poles} - \sum \text{finite zeros}}{\#\,\text{finite poles} - \#\,\text{finite zeros}}$$

$$= \frac{(-2-4-6)-(0)}{3-0} = -4$$

$$\theta_a = \frac{(2k+1)\pi}{\#\,\text{finite poles} - \#\,\text{finite zeros}}$$

$$= \frac{\pi}{3},\ \pi,\ \frac{5\pi}{3}$$



Refining the Root Locus

Real-axis breakaway and break-in points

$$\sum_{1}^{m} \frac{1}{\sigma + z_i} = \sum_{1}^{n} \frac{1}{\sigma + p_i}$$

jω-axis crossings

Using Routh-Hurwitz criterion: Forcing a row of zeros in the Routh table will yield the gain; going back one row to the even polynomial equation and solving for the roots yields the frequency at the jω-axis crossing.

Angles of departure and arrival

Assuming a point (on root locus) close to a *complex pole* or *zero*: The sum of angles drawn from all finite poles and zeros to this point is an odd multiple of 180°.

Example

Find the breakaway and break-in points for the root locus

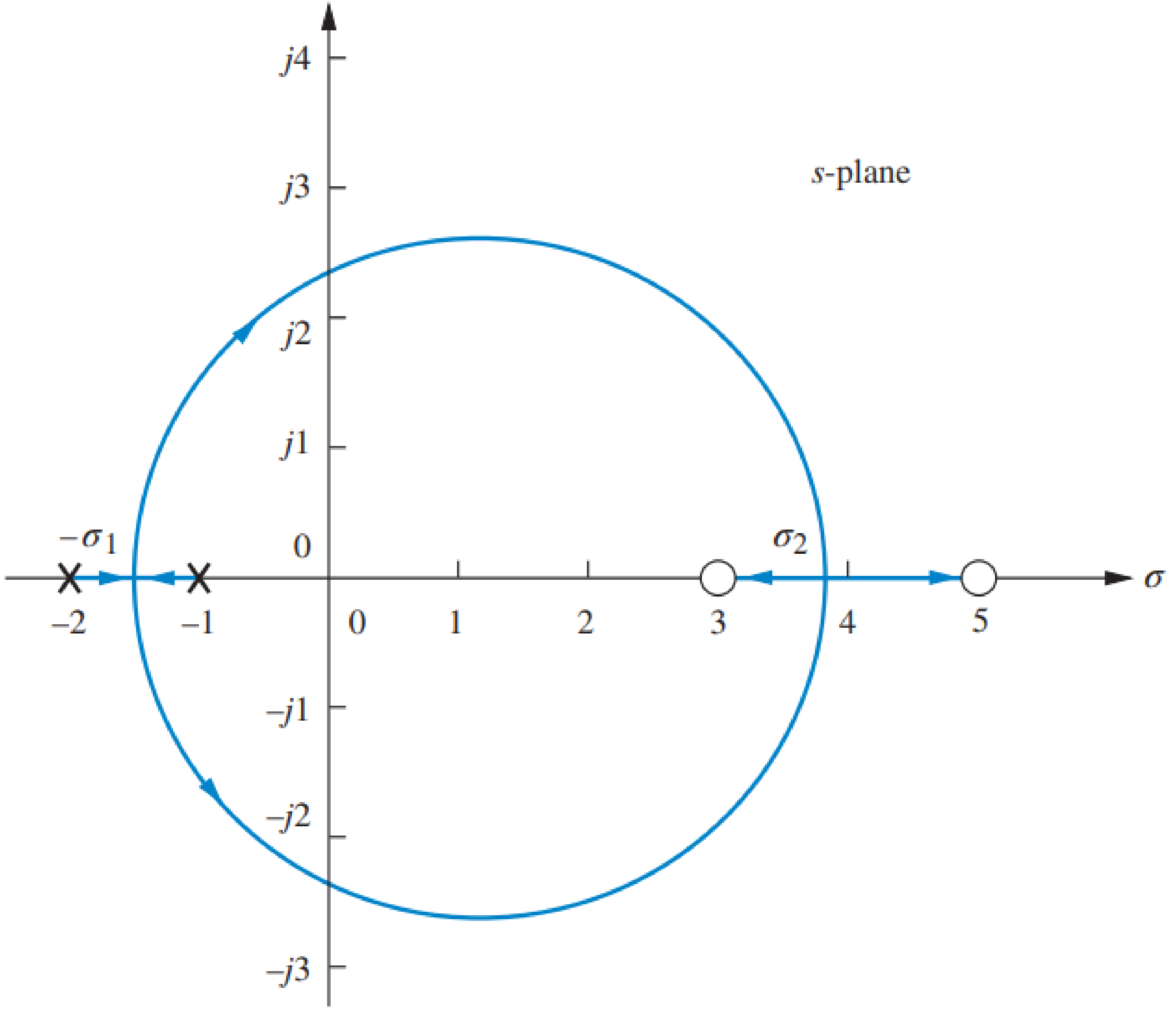
$$KG(s)H(s) = \frac{K(s-3)(s-5)}{(s+1)(s+2)}$$

$$\frac{1}{\sigma-3} + \frac{1}{\sigma-5} = \frac{1}{\sigma+1} + \frac{1}{\sigma+2}$$

Simplifying,

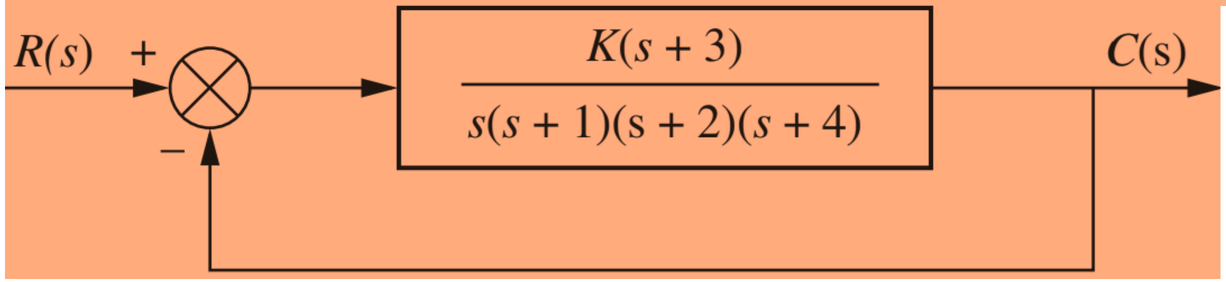
$$11\sigma^2 - 26\sigma - 61 = 0$$

$$\sigma = -1.45 \text{ and } 3.82$$



continued...

For the system, find the frequency and gain, K, for which the root locus crosses the imaginary axis. For what range of K is the system stable?



$$T(s) = \frac{K(s+3)}{s^4 + 7s^3 + 14s^2 + (8+K)s + 3K}$$

s^4	1	14	$3K$
s^3	7	$8+K$	
s^2	$90-K$	$21K$	
s^1	$\frac{-K^2-65K+720}{90-K}$		
s^0	$21K$		

In this table, for positive values of gain, only the s^1 row can yield a row of zeros.

$$-K^2 - 65K + 720 = 0$$

$$K = 9.65$$

Forming the even polynomial by using the s^2 row with $K = 9.65$,

$$(90-K)s^2 + 21K = 0$$

$$80.35s^2 + 202.7 = 0$$

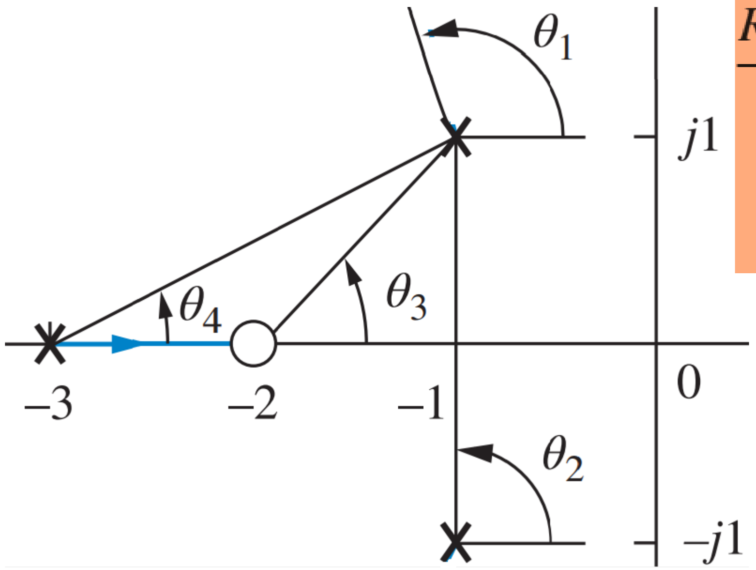
$$s = \pm j1.59$$

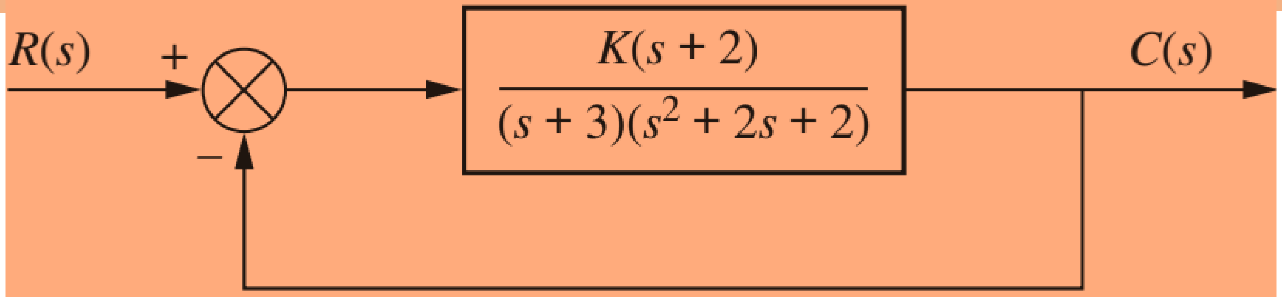
Thus the root locus crosses the jω-axis at $\pm j1.59$ at a gain of 9.65.

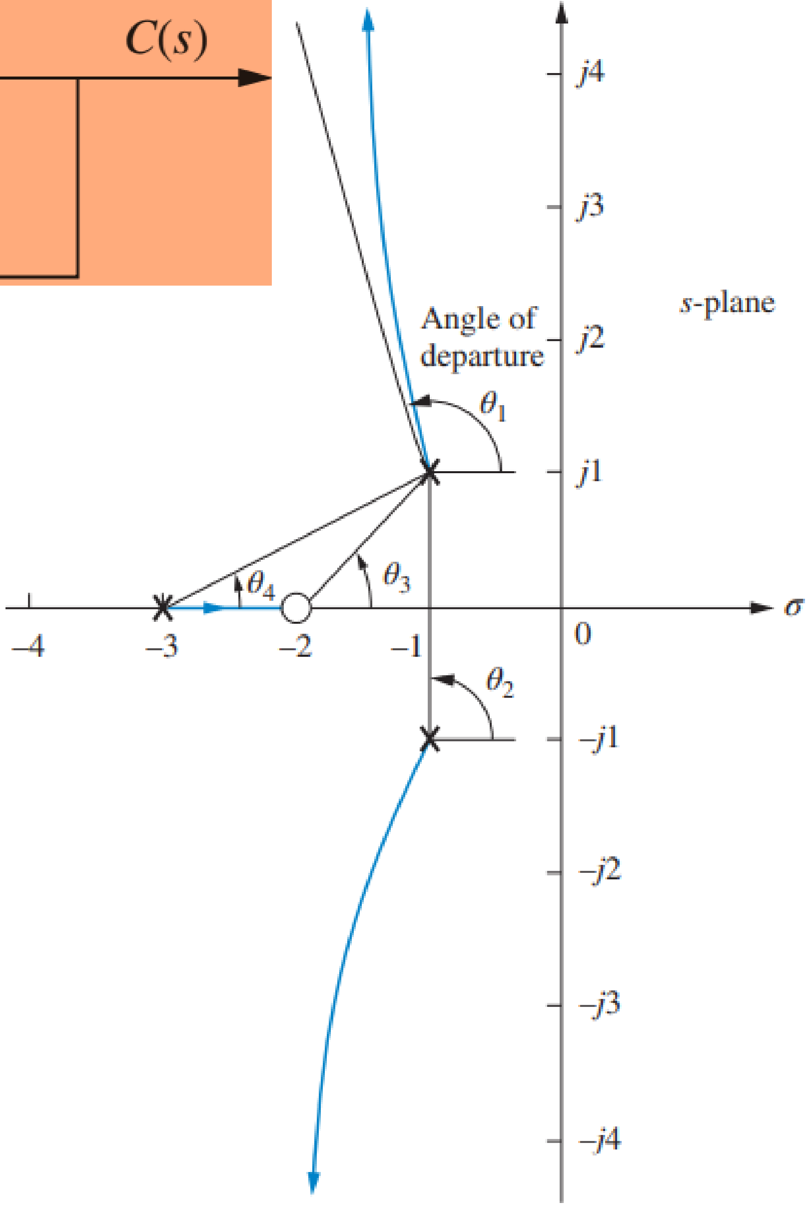
The system is stable for $0 \le K < 9.65$.

continued…

Find the angle of departure from the complex poles and sketch the root locus.







Complex poles at $-1 \pm j1$

We take $-1 + j1$

$$-\theta_1 - \theta_2 + \theta_3 - \theta_4 = 180^\circ$$

$$-\theta_1 - 90^\circ + \tan^{-1}\left(\frac{1}{1}\right) - \tan^{-1}\left(\frac{1}{2}\right) = 180^\circ$$

$$\theta_1 = -251.6^\circ = 108.4^\circ$$

continued…

a. Sketch the root locus.

b. Find the imaginary-axis crossing.

c. Find the gain, K, at the $j\omega$-axis crossing.

d. Find the break-in point.

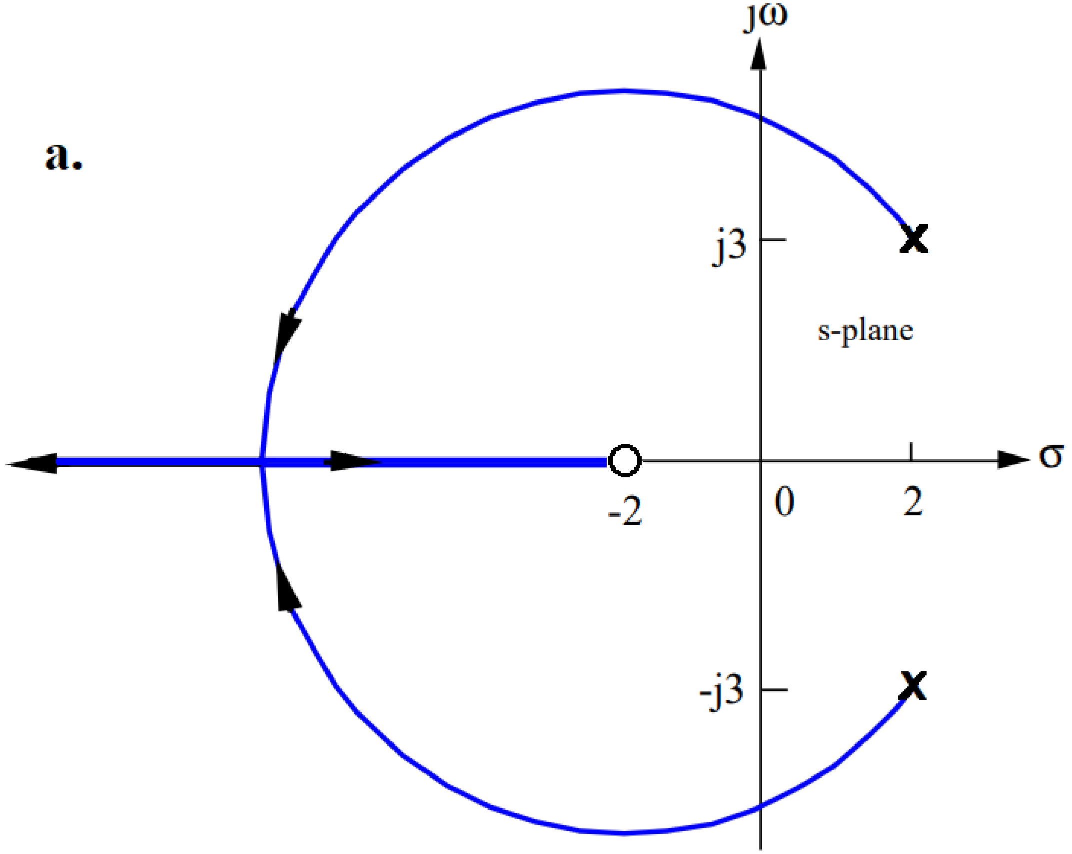
e. Find the angle of departure from the complex poles.

$$G(s) = \frac{K(s+2)}{(s^2 - 4s + 13)}$$

[Given a unity feedback system that has this forward transfer function]

- Poles at $2 \pm j3$
- Zeros at -2 and ∞

a.



b.

$$T(s) = \frac{G(s)}{1+G(s)}$$

$$= \frac{K(s+2)}{s^2 + (K-4)s + (2K+13)}$$

s^2	1	$2K+13$
s^1	K-4	0
s^0	$2K+13$	0

We get a row of zeros for $K = 4$.

continued...

From the s^2 row with $K = 4$,

$$s^2 + 21 = 0$$

$$\Rightarrow s = \pm j4.58$$

s^2	1	$2K$+13
s^1	K-4	0
s^0	$2K$+13	0

c. $K = 4$.

d.

$$\frac{1}{\sigma + 2} = \frac{1}{\sigma - 2 - j3} + \frac{1}{\sigma - 2 + j3}$$

$$\Rightarrow \frac{1}{\sigma + 2} = \frac{2\sigma - 4}{\sigma^2 - 4\sigma + 13}$$

$$\Rightarrow 2\sigma^2 - 8 = \sigma^2 - 4\sigma + 13$$

$$\Rightarrow \sigma^2 + 4\sigma - 21 = 0$$

$$\Rightarrow \sigma = -7, 3$$

But, 3 is unacceptable.

break-in point is at –7.

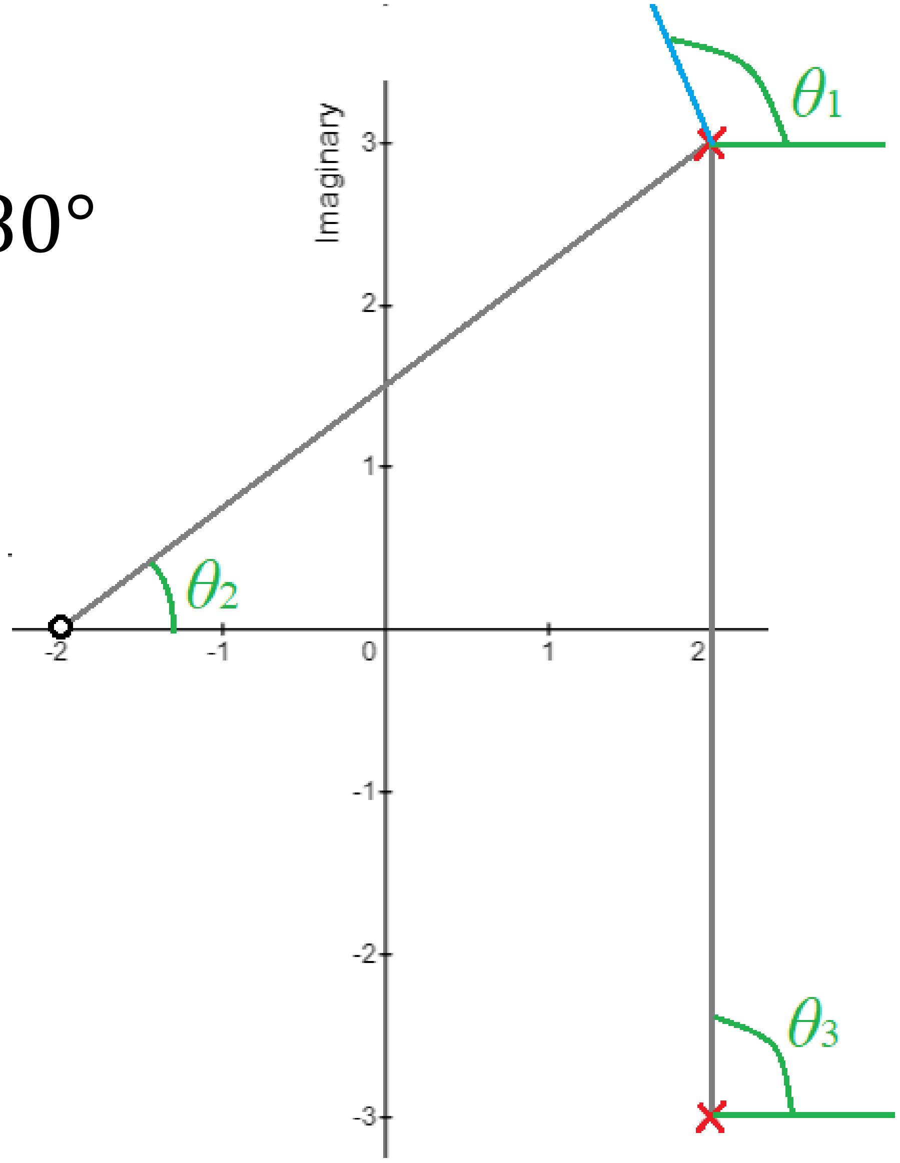
continued...

e.

$$-\theta_1 + \theta_2 - \theta_3 = 180°$$

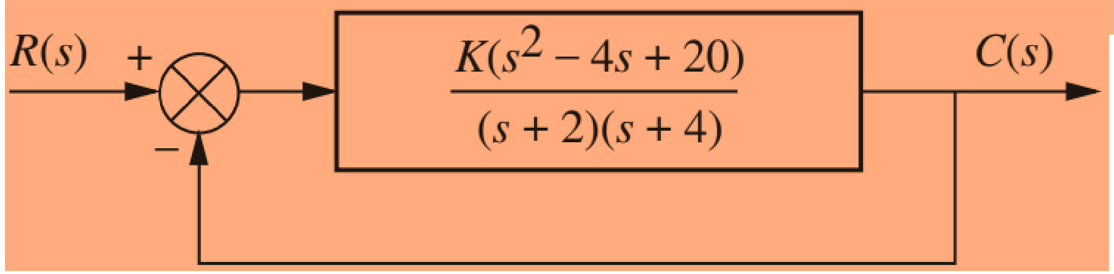
$$\Longrightarrow -\theta_1 + \tan^{-1}\left(\frac{3}{4}\right) - 90° = 180°$$

$$\Longrightarrow \theta_1 = -233.1° = 126.9°$$



continued…

Sketch the root locus for the system shown. Find the exact point and gain where the locus crosses the jω-axis. Find the breakaway point on the real axis. Find the angle of arrival. Find the range of K within which the system is stable.



Poles at -2 and -4.
Zeros at $2 \pm j4$.

$$T(s) = \frac{K(s^2 - 4s + 20)}{(1+K)s^2 + (6-4K)s + 8 + 20K}$$

<u>Routh Table:</u>

s^2	$1+K$	$8+20K$
s^1	$6-4K$	0
s^0	$\dfrac{-80K^2+88K+48}{6-4K}$	0

We get row of zeros for $K = \frac{6}{4} = 1.5$.

Now, from the s^2 row with $K = 1.5$,

$(1+1.5)s^2 + (8 + 20 \times 1.5) = 0$

$\Rightarrow 2.5s^2 + 38 = 0$

$\Rightarrow s = \pm j3.89$ [jω-axis crossing]

Now, for breakaway point,

$$\frac{1}{\sigma+2} + \frac{1}{\sigma+4} = \frac{1}{\sigma-2-j4} + \frac{1}{\sigma-2+j4}$$

$$\Rightarrow \frac{2\sigma+6}{\sigma^2+6\sigma+8} = \frac{2\sigma-4}{\sigma^2-4\sigma+20}$$

continued…

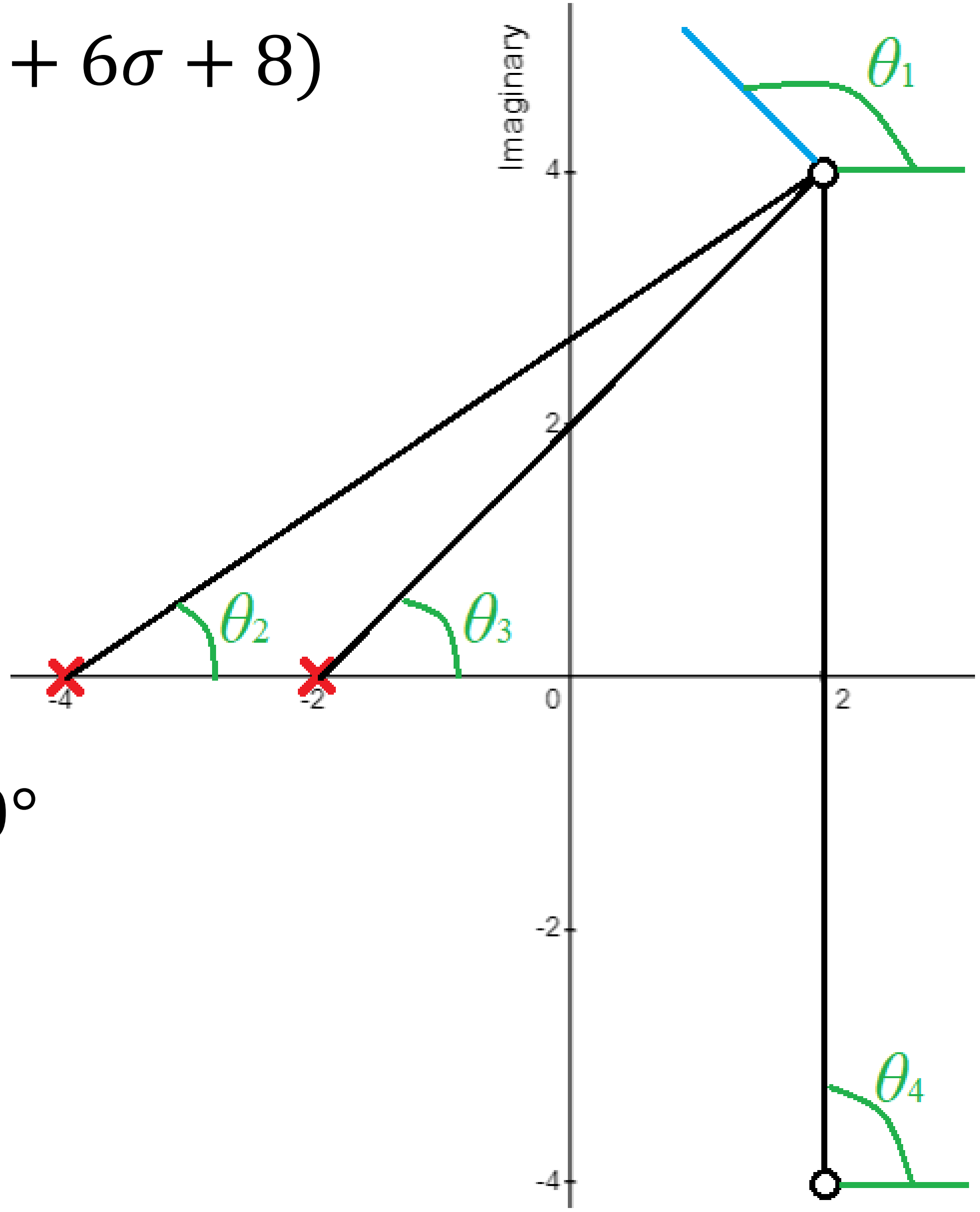
$$\Longrightarrow (2\sigma + 6)(\sigma^2 - 4\sigma + 20) = (2\sigma - 4)(\sigma^2 + 6\sigma + 8)$$
$$\Longrightarrow 10\sigma^2 - 24\sigma - 152 = 0$$
$$\Longrightarrow \sigma = -2.87, 5.27$$

But 5.27 is unacceptable.

So, the breakaway point is at -2.87.

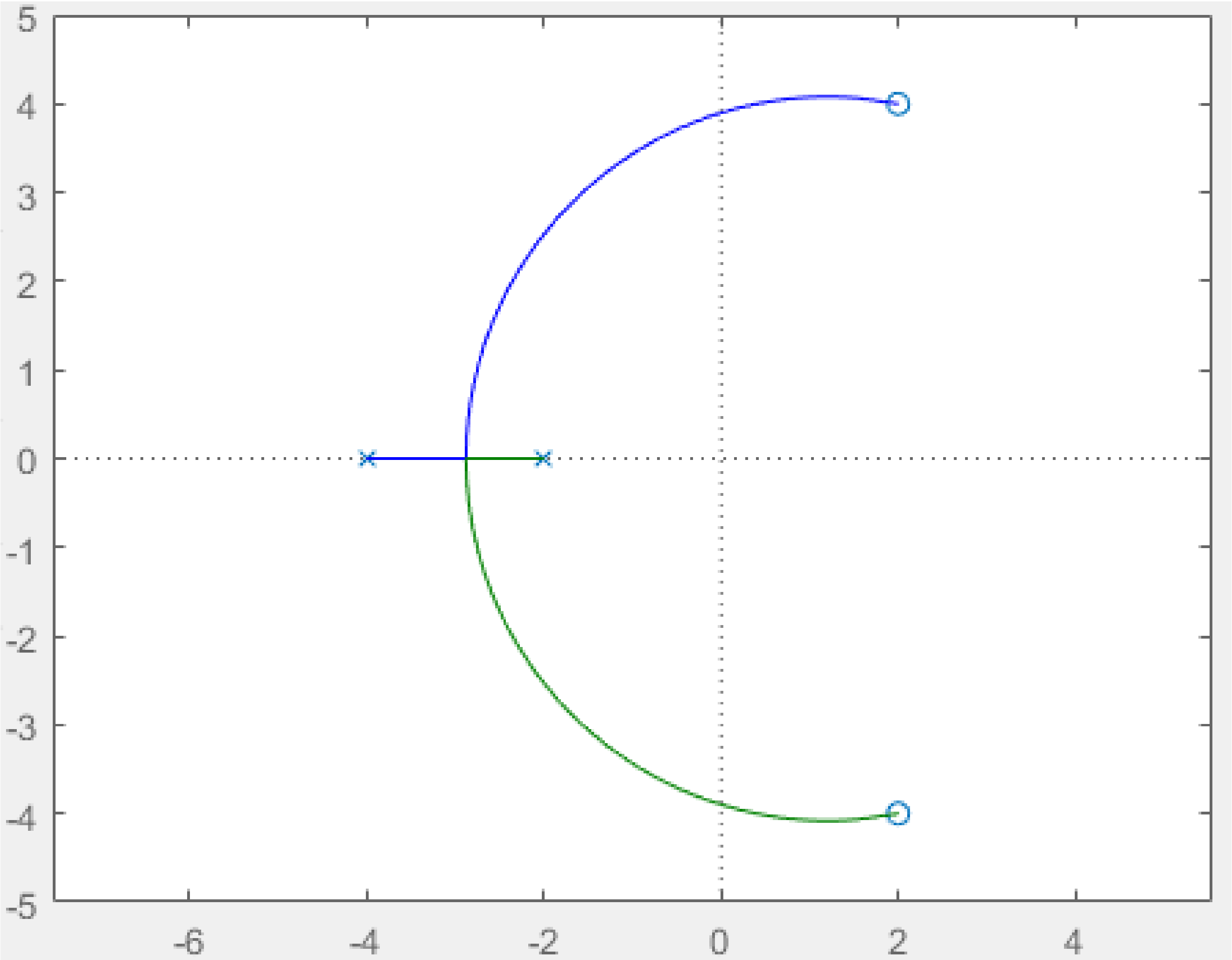
For angle of arrival,

$$\theta_1 - \theta_2 - \theta_3 + \theta_4 = 180°$$
$$\Longrightarrow \theta_1 - \tan^{-1}\left(\frac{4}{6}\right) - \tan^{-1}\left(\frac{4}{4}\right) + 90° = 180°$$
$$\Longrightarrow \theta_1 = 168.7°$$



continued…

This system is stable for $0 \leq K < 1.5$.



continued…

Sketch the root locus for a unity negative feedback system having the following forward transfer function:

$$G(s) = \frac{K\left(s+\frac{2}{3}\right)}{s^2(s+6)}$$

$$\sigma_a = \frac{\sum \text{finite poles} - \sum \text{finite zeros}}{\#\text{ finite poles} - \#\text{ finite zeros}}$$

$$= \frac{0+0-6-(-\frac{2}{3})}{3-1} = -2.67$$

$$\theta_a = \frac{(2k+1)\pi}{\#\text{ finite poles} - \#\text{ finite zeros}}$$

$$= \frac{\pi}{2}, \frac{3\pi}{2}$$

Now, for breakaway and break-in point,

$$\frac{1}{\sigma+0} + \frac{1}{\sigma+0} + \frac{1}{\sigma+6} = \frac{1}{\sigma+\frac{2}{3}}$$

$$\Rightarrow \frac{3\sigma+12}{\sigma^2+6\sigma} = \frac{1}{\sigma+\frac{2}{3}}$$

$$\Rightarrow 2\sigma^2+8\sigma+8=0$$

$$\Rightarrow \sigma = -2, -2$$

continued...