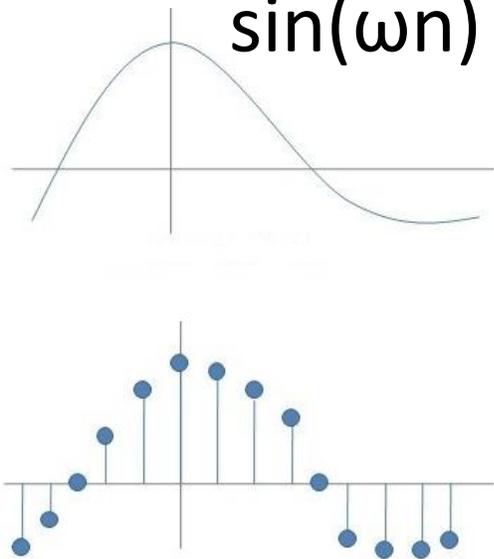# Digital Signal Processing I

EEE 313

Dept. of EEE, BUBT

Signal

"Stream of information that varies with some variables"

- Continuous Signal: $\sin(\Omega t)$

- Discrete Signal: $\sin(\omega n)$

Types by dimension:

➢ 1D: Audio, ECG

➢ 2D: Photograph

➢ 3D: Video

All natural signals are analog.

When an analog signal is sampled, quantized and coded, it becomes a digital signal.

Sampling

$x(t) = A \cos(\Omega t)$

After sampling with T_S interval,

$$x(nT_S) = A \cos(\Omega n T_S)$$
$$= A \cos(2\pi F n T_S)$$
$$= A \cos(2\pi n F / F_S)$$
$$= A \cos(2\pi f n)$$

- Ω = Analog Angular Frequency (rad/second) = $2\pi F$

- F = Analog Frequency (cycle/second)

- F_S = Sampling Frequency (sample/second) = $1 / T_S$

- f = Discrete Time Frequency (cycle/sample) = F / F_S

Example: f = 1/70 means "70 samples are collected in 1 cycle".

Nyquist's Sampling Theorem

"A bandlimited continuous-time signal can be perfectly reconstructed from its samples if the waveform is sampled over twice as fast as its highest frequency component."

For lossless reconstruction,

$F_s \geq 2F$

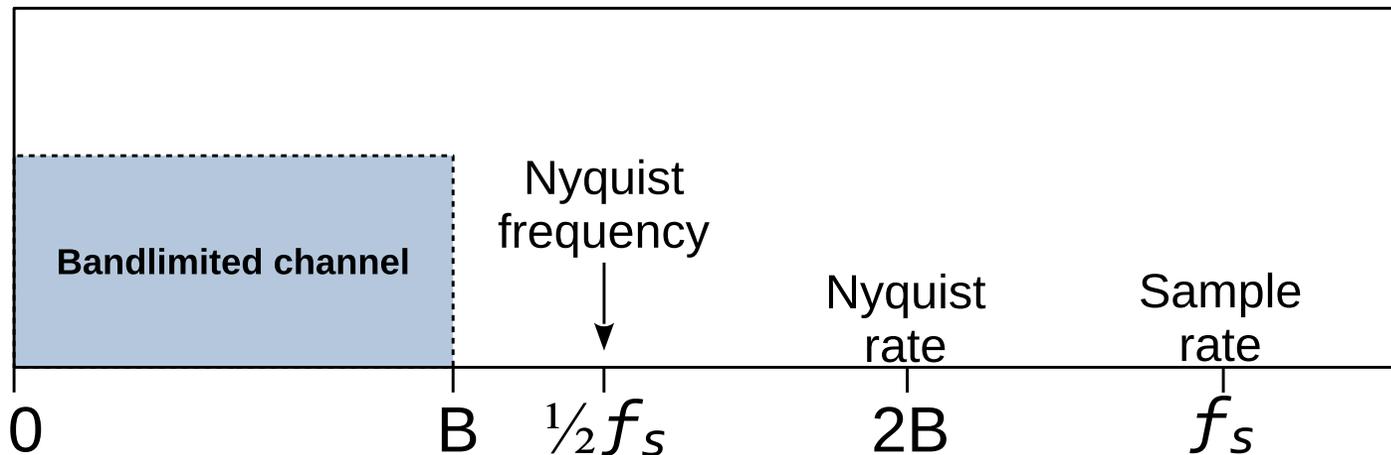or, $F / F_s \leq 1 / 2$

or, $f \leq 1 / 2$

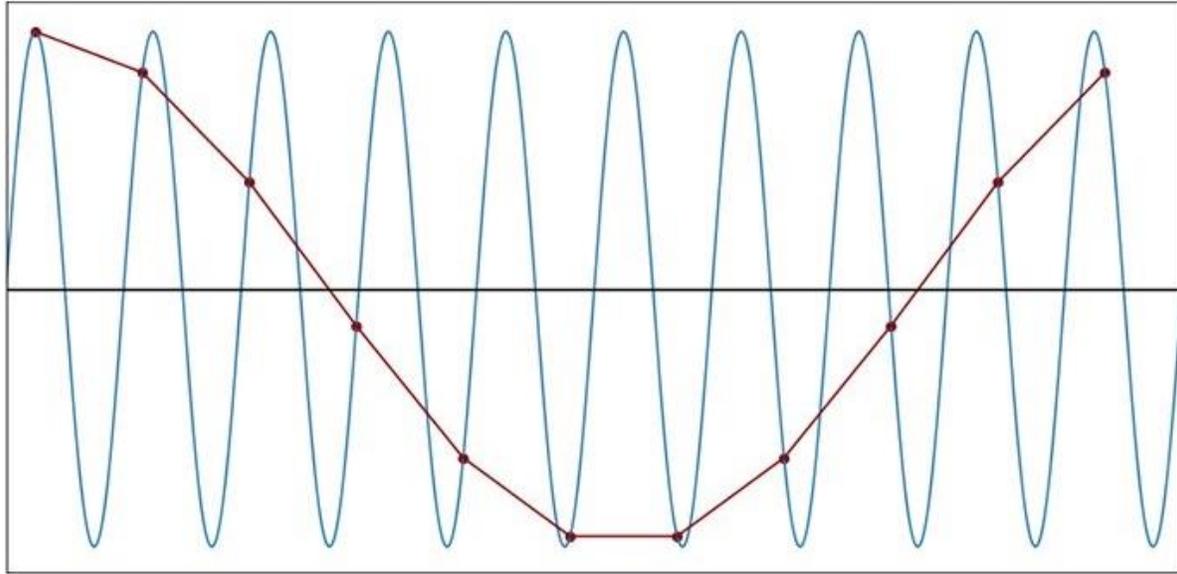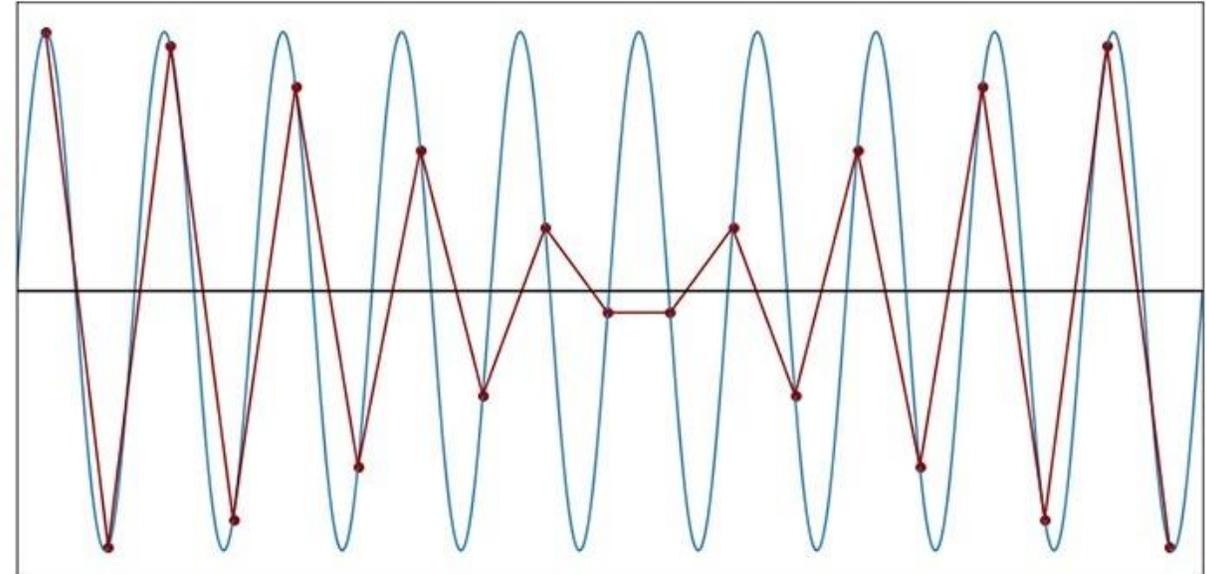So, f_{max} can be $1 / 2$.

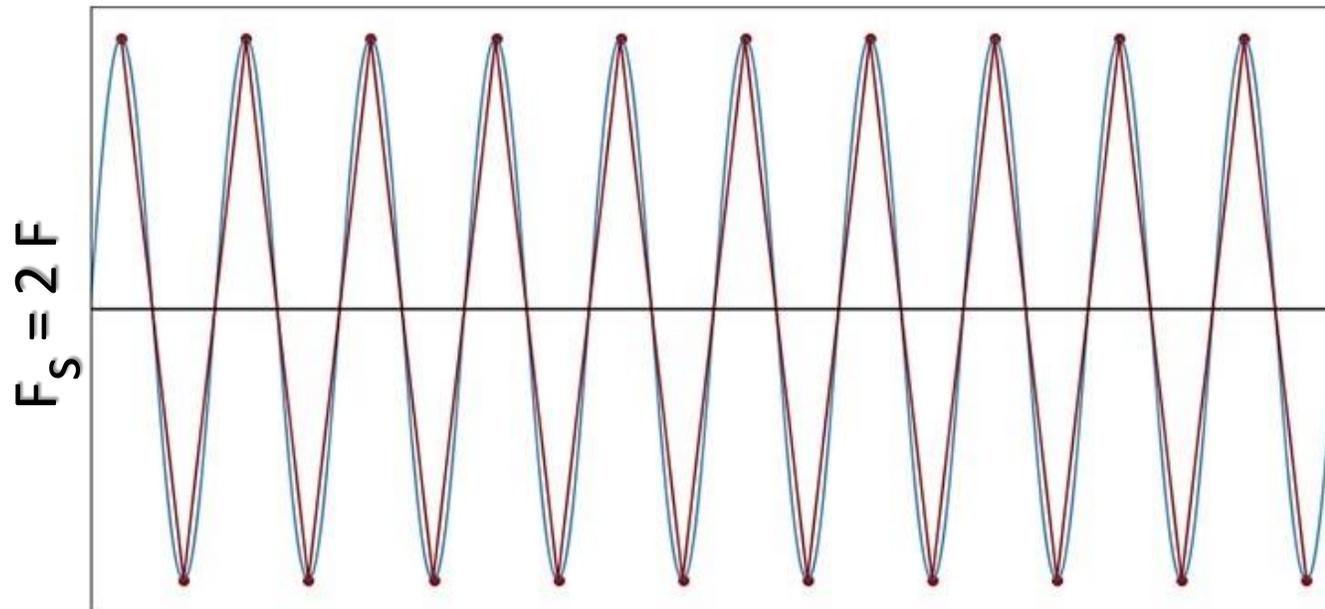*Note:* Smaller f is better. Because smaller f means: more samples are taken in each cycle.

	Nyquist frequency	Nyquist rate	Sample rate
Bandlimited channel			

0   B $\frac{1}{2}f_s$ 2B f_s

Nyquist (continued)…

$F_S = 1.1\ F$

$F_S = 1.9\ F$

$F_S = 2\ F$

Periodicity

$$x[n + N] = A\cos\big(\omega(n + N)\big)$$

Previously, period was time. But here, period is sample.

Also, ω = 2πf = 2πk/N, and k is integer.

$$x[n + N] = A\cos(\omega n + \omega N)$$

$$= A\cos\left(\omega n + \frac{2\pi k}{N} N\right)$$

$$= A\cos(\omega n + 2\pi k)$$

$$= A\cos(\omega n)$$

$$= x[n]$$

Note: f (= k/N) must be rational number.

Question: Is "$15\cos(\sqrt{5}\pi n)$" a valid representation of discrete-time signal? Also find for "$15\cos(10n)$".

➤ $15\cos(\sqrt{5}\pi n) = 15\cos\left(2\pi\frac{\sqrt{5}}{2}n\right)$

Here, $f = \frac{\sqrt{5}}{2}$, which can not be a rational number.

So, NOT a valid representation.

➤ $15\cos(10n) = 15\cos\left(2\pi\frac{5}{\pi}n\right)$

Here, $f = \frac{5}{\pi}$, which can not be a rational number (because π is irrational)

So, NOT a valid representation.

Question: $x(t) = 20 \cos(\Omega_1 t) + 30 \cos(\Omega_2 t)$, where $F_1 = 2000$ Hz and $F_2 = 3000$ Hz. What is the minimum sampling frequency for lossless reconstruction? Find the expression of $x[n]$ for 4000 Hz sampling frequency.

$F_{S(min)}$ must be twice the maximum frequency component in $x(t)$.
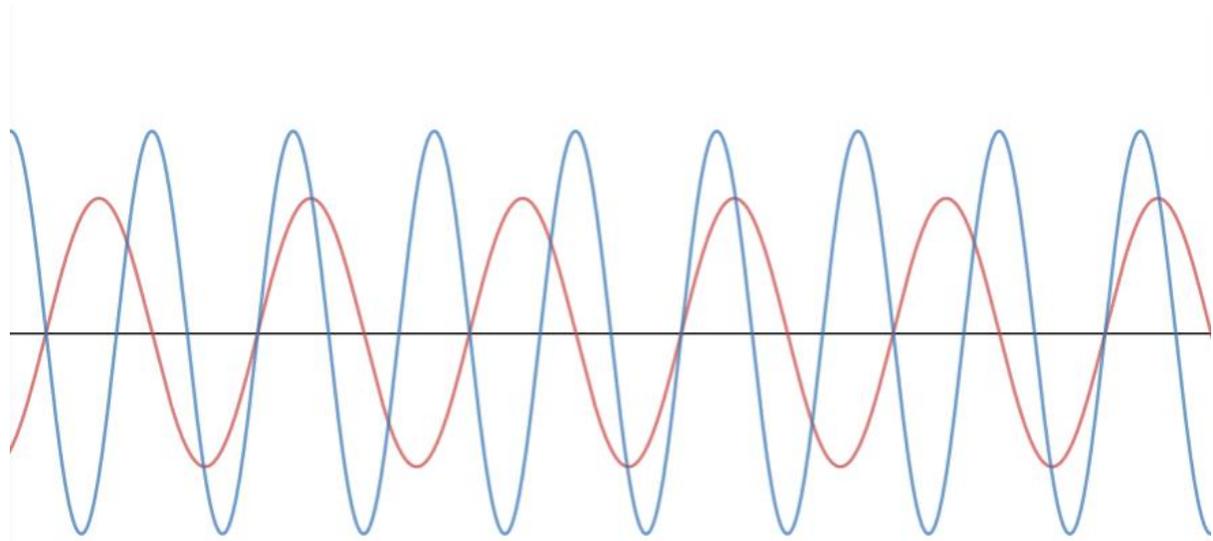
So, it should be 6000 Hz.

Now,
$$x(t) = 20 \cos(\Omega_1 t) + 30 \cos(\Omega_2 t)$$
$$= 20 \cos(2\pi F_1 t) + 30 \cos(2\pi F_2 t)$$
$$= 20 \cos(2\pi 2000 t) + 30 \cos(2\pi 3000 t)$$

So,
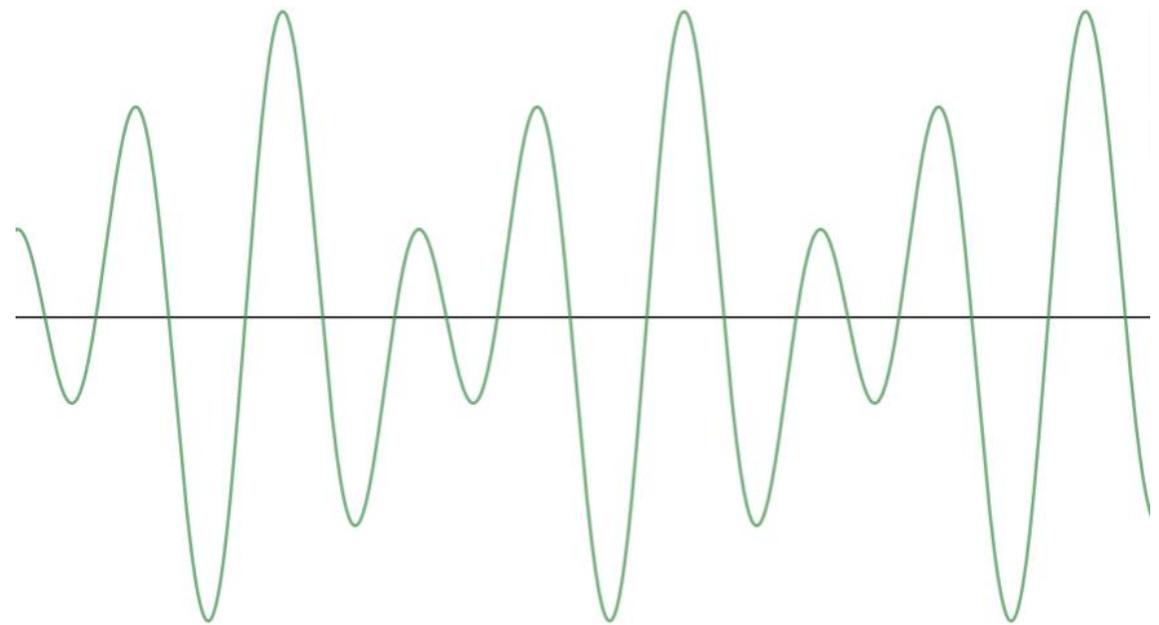$$x[n] = 20 \cos(2\pi \, {}^{2000}/_{4000} \, n) + 30 \cos(2\pi \, {}^{3000}/_{4000} \, n)$$
$$= 20 \cos\left(2\pi \frac{1}{2} n\right) + 30 \cos\left(2\pi \frac{3}{4} n\right)$$

Red: $20\cos(2\pi 2000t)$
Blue: $30\cos(2\pi 3000t)$

Green:
$20\cos(2\pi 2000t) + 30\cos(2\pi 3000t)$

Time-Shifting of DT Sequence

Delay

y[n] = x[n - d], where d is integer.

Let, x[n] = [... 2 5 $\underline{7}$ 6 9 1 ...]

y[n] = x[n-1]

y[-1] = x[-1-1] = x[-2] = 2

y[0] = x[0-1] = x[-1] = 5

y[1] = x[1-1] = x[0] = 7

y[2] = x[2-1] = x[1] = 6   *and so on.*

So, y[n] = [... 2 $\underline{5}$ 7 6 9 1 ...]

Advance

y[n] = x[n + a], where a is integer.

Let, x[n] = [... 2 5 $\underline{7}$ 6 9 1 ...]

y[n] = x[n+1]

y[-1] = x[-1+1] = x[0] = 7

y[0] = x[0+1] = x[1] = 6

y[1] = x[1+1] = x[2] = 9

y[2] = x[2+1] = x[3] = 1   *and so on.*

So, y[n] = [... 2 5 7 $\underline{6}$ 9 1 ...]

Basic Properties of Discrete-Time System

(1) <u>Memory</u>

A system is memoryless if output depends only on present value.

Let, $y[n] = T\{x[n]\} = x^2[n]$, where T{x[n]} means a transform in a system.

Now, $y[5] = x^2[5]$, $y[17] = x^2[17]$ and so on.

It is memoryless.

Now let, $y[n] = T\{x[n]\} = x[n] + x[n-2]$.

Thus, $y[5] = x[5] + x[3]$, it depends on past value.

It needs memory.

Now let, $y[n] = T\{x[n]\} = x[n] + x[n+3]$.

Thus, $y[5] = x[5] + x[8]$, it depends on future value.

It needs memory.

<u>Question:</u> x[n] = [9 6 5 1 4 0 7 0 3 …]; $y[n] = T\{x[n]\} = x[n] - x[n+1]$.
Find y[-1], y[0], y[1], y[2], y[3]. Is this system memoryless?

y[-1] = x[-1] − x[0] = 1 − 4 = -3
y[0] = x[0] − x[1] = 4 − 0 = 4
y[1] = x[1] − x[2] = 0 − 7 = -7
y[2] = x[2] − x[3] = 7 − 0 = 7
y[3] = x[3] − x[4] = 0 − 3 = -3

The system depends on future values. So, not memoryless.

(2) Linearity

A system is linear if it satisfies additivity & homogeneity of degree one.

(a) Additivity: For every pair of signal $x_1[n]$ and $x_2[n]$,
$$T\{x_1[n] + x_2[n]\} = T\{x_1[n]\} + T\{x_2[n]\}$$

(b) Homogeneity of degree one: For every signal $x[n]$ and scalar a,
$$T\{ax[n]\} = a\,T\{x[n]\}$$

Combining these two, $T\{ax_1[n] + bx_2[n]\} = a\,T\{x_1[n]\} + bT\{x_2[n]\}$
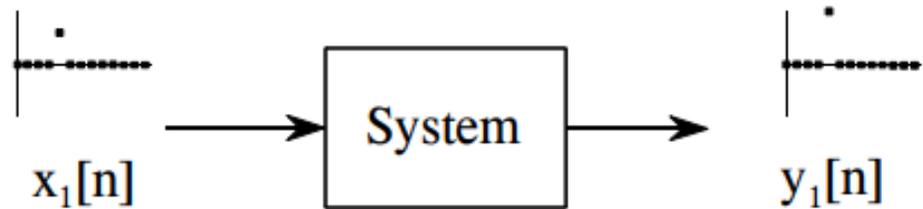
For example, let, $y[n] = T\{x[n]\} = x^2[n]$.

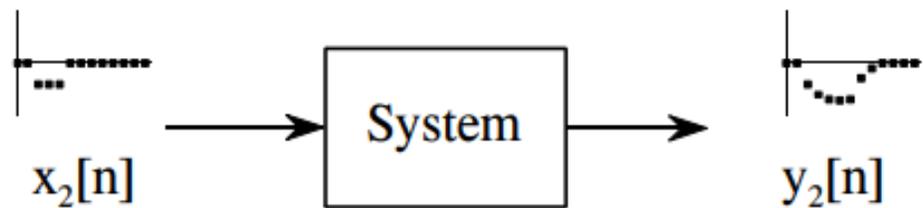Now, $(x_1[n] + x_2[n])^2 \neq (x_1{}^2[n] + x_2{}^2[n])$.
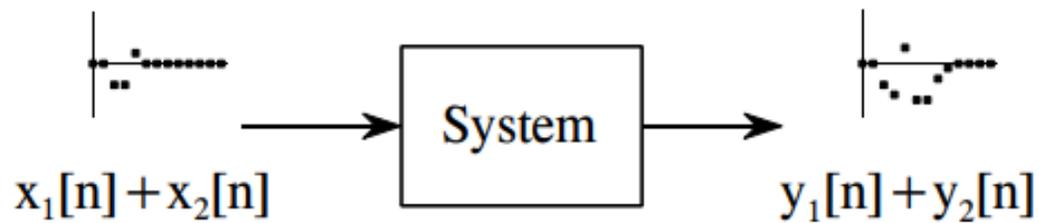
So, it's non-linear.

Additivity

IF

$x_1[n]$ → System → $y_1[n]$

AND IF

$x_2[n]$ → System → $y_2[n]$

THEN

$x_1[n] + x_2[n]$ → System → $y_1[n] + y_2[n]$

Homogeneity

IF

$x[n]$ → System → $y[n]$
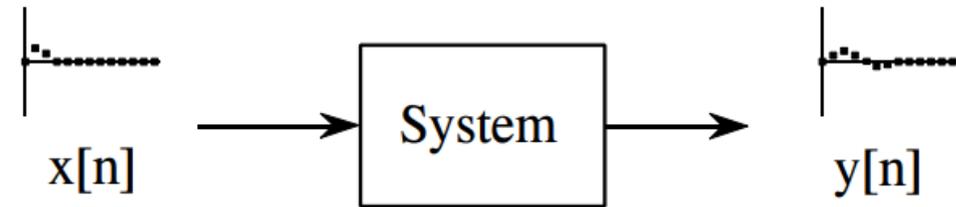
THEN

$k\,x[n]$ → System → $k\,y[n]$

continued…

<u>Question:</u> $x_1[n] = [\underline{2} \ 0 \ 4 \ 6 \ …]$; $x_2[n] = [\underline{1} \ 3 \ 2 \ 7 \ …]$. Using these sequences, show if linearity holds for the following system: $y[n] = T\{x[n]\} = x^2[n]$.

$$y_1[n] = T\{ax_1[n] + bx_2[n]\}$$

$$= \left[\ \underline{(2+1)^2} \quad (0+3)^2 \quad (4+2)^2 \quad (6+7)^2 \ … \right]$$

$$= [\ \underline{9} \quad 9 \quad 36 \quad 169 \ …]$$

$$y_2[n] = a \ T\{x_1[n]\} + bT\{x_2[n]\}$$

$$= \left[\ \underline{(2^2 + 1^2)} \quad (0^2 + 3^2) \quad (4^2 + 2^2) \quad (6^2 + 7^2) \ … \right]$$

$$= \left[\ \underline{5} \quad 9 \quad 20 \quad 85 \ … \right]$$

So, $y_1 \neq y_2$

The system is non-linear.

continued...

Question: $x_1[n] = [\ \underline{2}\ 0\ 4\ 6\ ...]$; $x_2[n] = [\ \underline{1}\ 3\ 2\ 7\ ...]$. Using these sequences, show if linearity holds for the following system: $y[n] = T\{x[n]\} = 2x[n]$.

$$y_1[n] = T\{ax_1[n] + bx_2[n]\}$$

$$= \left[\ \underline{2(2+1)}\ \ 2(0+3)\ \ 2(4+2)\ \ 2(6+7)\ ...\right]$$

$$= \left[\ \underline{6}\ \ 6\ \ 12\ \ 26\ ...\right]$$

$$y_2[n] = a\,T\{x_1[n]\} + bT\{x_2[n]\}$$

$$= [\ \underline{(4+2)}\ \ (0+6)\ \ (8+4)\ \ (12+14)\ ...]$$

$$= [\ \underline{6}\ \ 6\ \ 12\ \ 26\ ...]$$

So, $y_1 = y_2$

The system is linear.

continued…

(3) Time-Invariance

A system is time-invariant if for all n_d, the input with values $x_1[n] = x[n - n_d]$ produces output with values $y_1[n] = y[n - n_d]$.
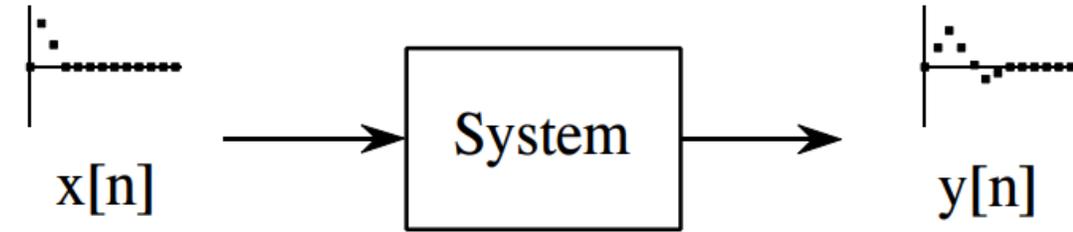
In other words:

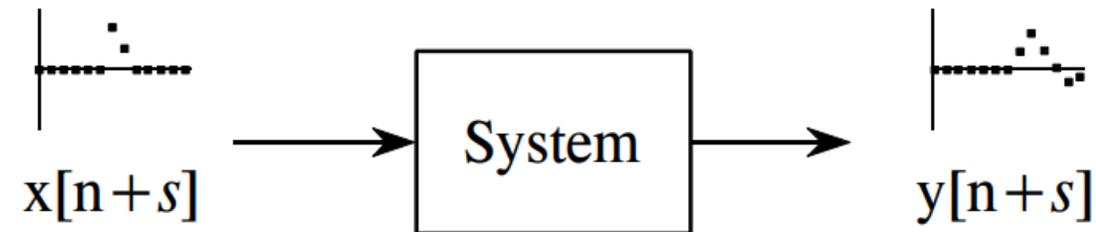   (a) Get your output with no input delay, then delay the output.
    (b) Delay the input, then get output.

If both cases are matched, the system is time-invariant.

IF

x[n] → System → y[n]

THEN

x[n+s] → System → y[n+s]

continued…

Let, $y[n] = T\{x[n]\} = x[2n]$

x[n] = [… a b c <u>d</u> e f g h …]

y[n] = [… b <u>d</u> f h …]   ; y[0]=x[0], y[-1]=x[-2], y[1]=x[2], y[2]=x[4]

$y_1[n]$ = y[n-1] = [… <u>b</u> d f h …]   [output is delayed]; $y_1[0]$=y[-1], $y_1[1]$=y[0], $y_1[2]$=y[1]

x[n-1] = [… a b <u>c</u> d e f g h …]   [input is delayed]

$y_2[n]$ = [… a <u>c</u> e g …]   [output for delayed input]

Here, $y_1[n] \neq y_2[n]$.

So, it's time-variant.

continued…

Let, $y[n] = T\{x[n]\} = x[3n]$

x[n] = [… 2 4 3 8 6 4 <u style="color:red">2</u> 1 4 9 5 9 7 …]

y[n] = [… 2 8 <u style="color:red">2</u> 9 7 …]   ; y[0]=x[0], y[-1]=x[-3], y[1]=x[3], y[2]=x[6]

$y_1[n]$ = y[n-1] = [… 2 <u style="color:red">8</u> 2 9 7…]   [output is delayed]; y_1[-1]=y[-2], y_1[0]=y[-1]

x[n-1] = [… 2 4 3 8 6 <u style="color:red">4</u> 2 1 4 9 5 9 7 …] [input is delayed]

$y_2[n]$ = [… 3 <u style="color:red">4</u> 4 9 …]   [output for delayed input]

Here, $y_1[n] \neq y_2[n]$.

So, it's time-variant.

Let, $y[n] = T\{x[n]\} = 5x[n]$

x[n] = [... 4 1 2 5 0 1 ...]

y[n] = [... 20 5 10 25 0 5 ...]

$y_1[n]$ = y[n-1] = [... 20 5 10 25 0 5 ...]   [output is delayed]; $y_1[0]$=y[-1], $y_1[1]$=y[0]

x[n-1] = [... 4 1 2 5 0 1 ...]   [input is delayed]

$y_2[n]$ = [... 20 5 10 25 0 5 ...]   [output for delayed input]

Here, $y_1[n] = y_2[n]$.

So, it's time-invariant.

Linear Time-Invariant (LTI) System

Unit impulse, $\quad \delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$

Now let, $\quad x[n] = \begin{bmatrix} \underline{5} & 7 & 9 & 2 & 4 & 1 & \dots \end{bmatrix}$
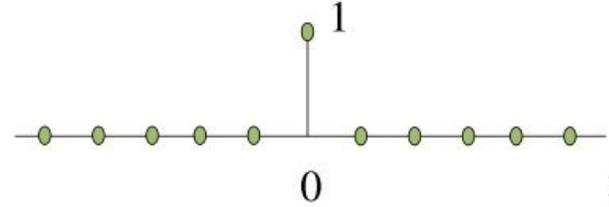
$$= 5\delta[n] + 7\delta[n-1] + 9\delta[n-2] + \cdots$$

$$= \sum_{k=-\infty}^{k=\infty} x[k]\,\delta[n-k]$$

Now,

$$y[n] = T\{x[n]\} = T\left\{ \sum_{k=-\infty}^{k=\infty} x[k]\,\delta[n-k] \right\}$$

continued...

If linearity holds, then

$$y[n] = T\left\{\sum_{k=-\infty}^{k=\infty} x[k]\,\delta[n-k]\right\} = \sum_{k=-\infty}^{k=\infty} x[k]\,T\{\delta[n-k]\}$$

And, for time-invariance, the impulse response,

$$h[n] = T\{\delta[n]\}$$
$$h[n-k] = T\{\delta[n-k]\}$$

So, for LTI system,

$$y[n] = \sum_{k=-\infty}^{k=\infty} x[k]\,h[n-k] = \sum_{k=-\infty}^{k=\infty} h[k]\,x[n-k]$$

Convolution Sum Formula

$$y[n] = \sum_{k=-\infty}^{k=\infty} x[k]\, h[n-k] = \sum_{k=-\infty}^{k=\infty} h[k]\, x[n-k]$$

Using convolution sum, we can predict output for any kind of input without physically putting on the equipment.

Some properties of convolution:

- Commutative: $x[n] * h[n] = h[n] * x[n]$
- Associative: $(x[n] * h[n]) * w[n] = x[n] * (h[n] * w[n])$
- Distributive: $x[n] * (h[n] + w[n]) = x[n] * h[n] + x[n] * w[n]$

(4) Stability

A system is stable if bounded input produces bounded output. [BIBO]

$$|x[n]| \leq B_x \; ; |y[n]| \leq B_y$$

We know, $y[n] = \sum_{k=-\infty}^{k=\infty} x[k]\, h[n-k]$ for LTI system.

For stability of LTI system,

$$|y[n]| \leq \sum_{k=-\infty}^{k=\infty} |x[k]|\, |h[n-k]|$$

$$\Longrightarrow |y[n]| \leq B_x \sum_{k=-\infty}^{k=\infty} |h[n-k]|$$

So, $\sum_{n=-\infty}^{n=\infty} |h[n]| < \infty$ (Necessary & Sufficient condition)

This means, impulse response must be absolutely summable.

Question: Find if the following systems are stable or not:

- y[n] = x[n] – x[n+1]

    If input is bounded, then difference between bounded values will also be bounded. So, the system is stable.

- y[n] = x[n] + 7

    If input is bounded, then addition of bounded values will also be bounded. So, the system is stable.

- y[n] = log (x[n])

    If 0 is found anywhere in the sequence x[n], then output becomes log(0) = $-\infty$. So, the system is unstable.

- y[n] = n x[n]

    Even if input is bounded, we don't know the boundary of n. So, the system is unstable.

(5) Causality

A system is causal if output depends on present and/or past input.

$y[n] = 4x[n]$	System is causal.
$y[n] = 2x[n-1]$	System is causal.
$y[n] = 7x[n] + 5x[n-1]$	System is causal.
$y[n] = 7x[n] + 5x[n+1]$	System is non-causal.
$y[n] = 5x[n+1]$	System is anti-causal & non-causal.

Note:
1. All anti-causal systems are non-causal, but not the opposite.
2. All memoryless systems are causal.
3. Real-time system can not be non-causal.

continued...

We know, $y[n] = \sum_{k=-\infty}^{k=\infty} h[k]\, x[n-k]$ for LTI system.

or, $y[n] = \cdots + h[-2]\, x[n+2] + h[-1]\, x[n+1] + h[0]\, x[n] +$
$h[1]\, x[n-1] + h[2]\, x[n-2] + \cdots$

For casual system, future values don't exist.

So, h[-1] = h[-2] = h[-3] = ... = 0

Simply, for any LTI system to be causal:

h[k] = 0, when k < 0

Question: x[n]=[1 2 3], h[n]=[4 5 6]. Find the convolution.

y[0] = 4

y[1] = 5+8 = 13

y[2] = 6+10+12 = 28

y[3] = 12+15 = 27

y[4] = 18

So, y[n] = [4 13 28 27 18]

		1	2	3			
6	5	4					
		1	2	3			
	6	5	4				
		1	2	3			
		6	5	4			
		1	2	3			
			6	5	4		
		1	2	3			
				6	5	4	

Question: x[n]=[1 2 **3** 4 5], h[n]=[4 **5** 6]. Find the convolution.

y[0] = 12+15+16 = 43

y[1] = 18+20+20 = 58

y[2] = 24+25 = 49

y[3] = 30

y[-1] = 6+10+12 = 28

y[-2] = 5+8 = 13

y[-3] = 4

So, y[n] = [4 13 28 **43** 58 49 30]
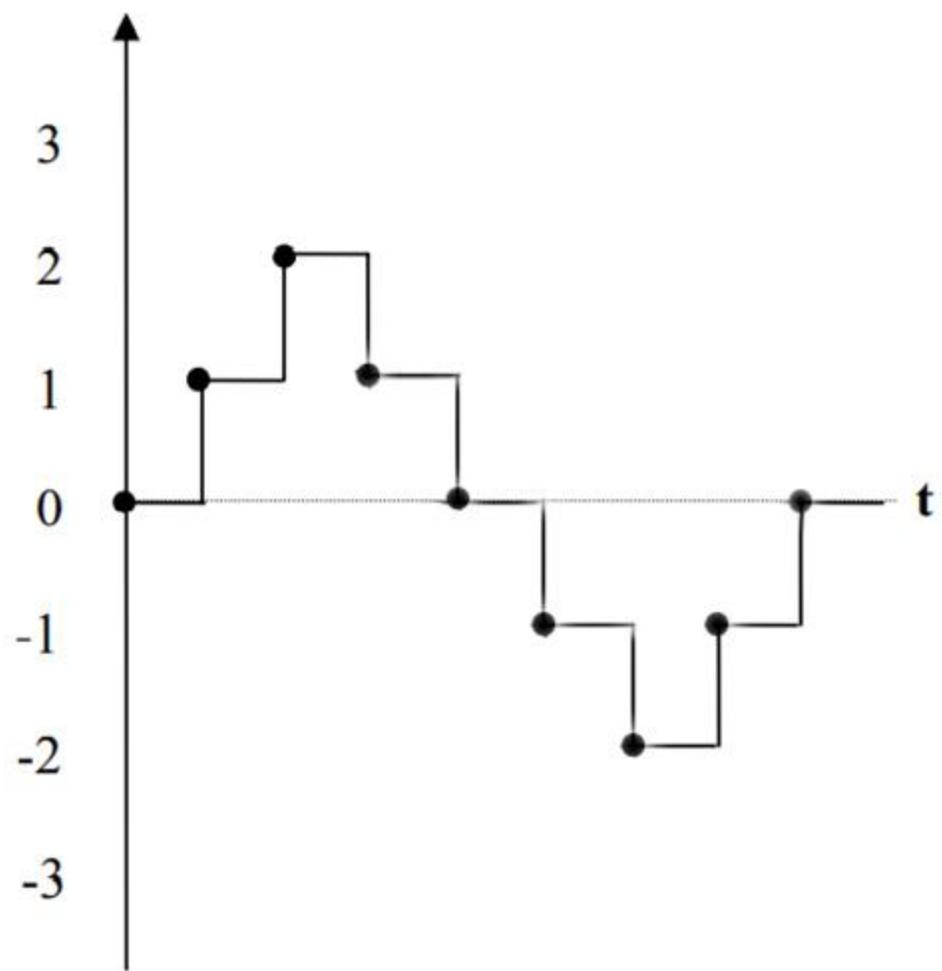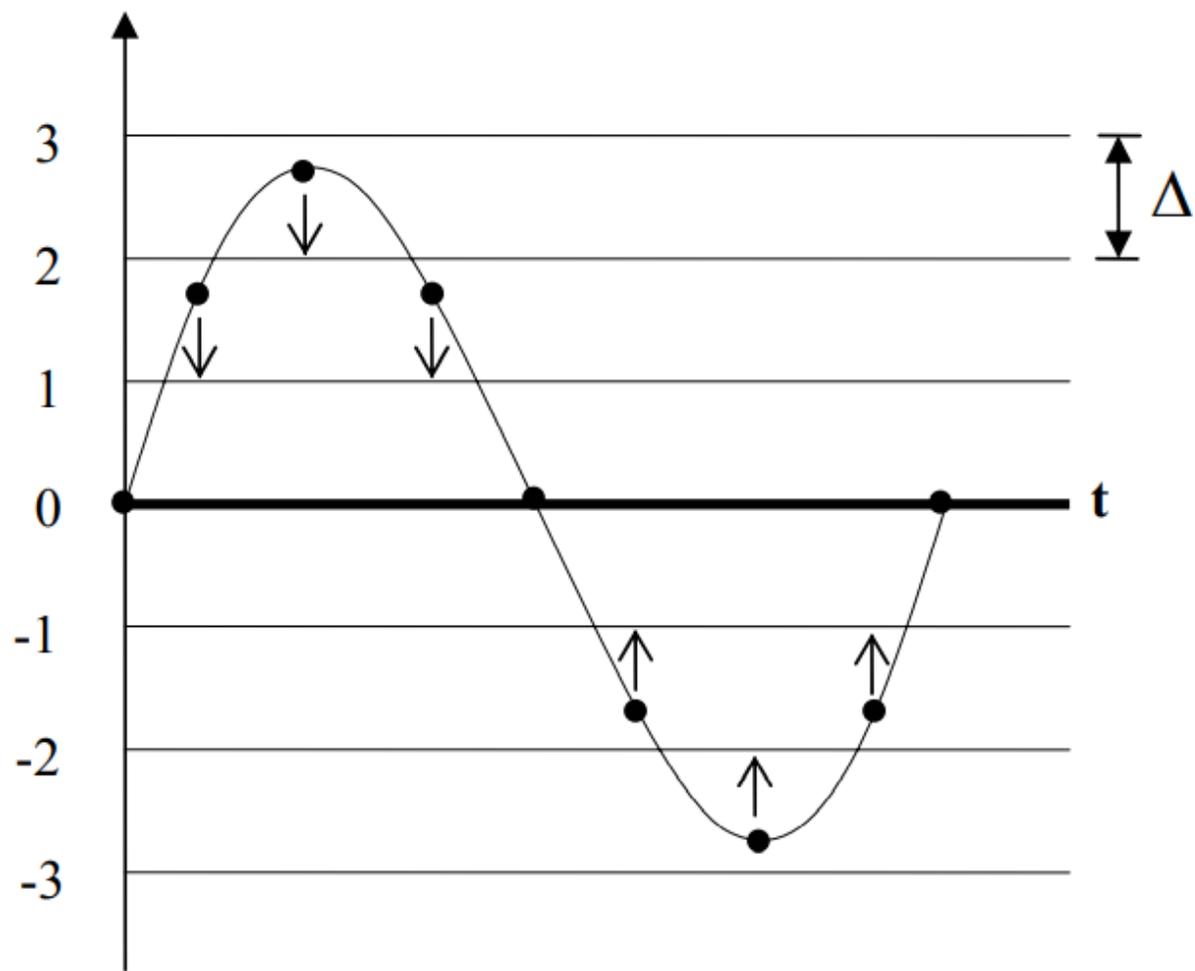
Convolution (graph)

Quantization

"Process of mapping continuous infinite values to a smaller set of discrete finite values"
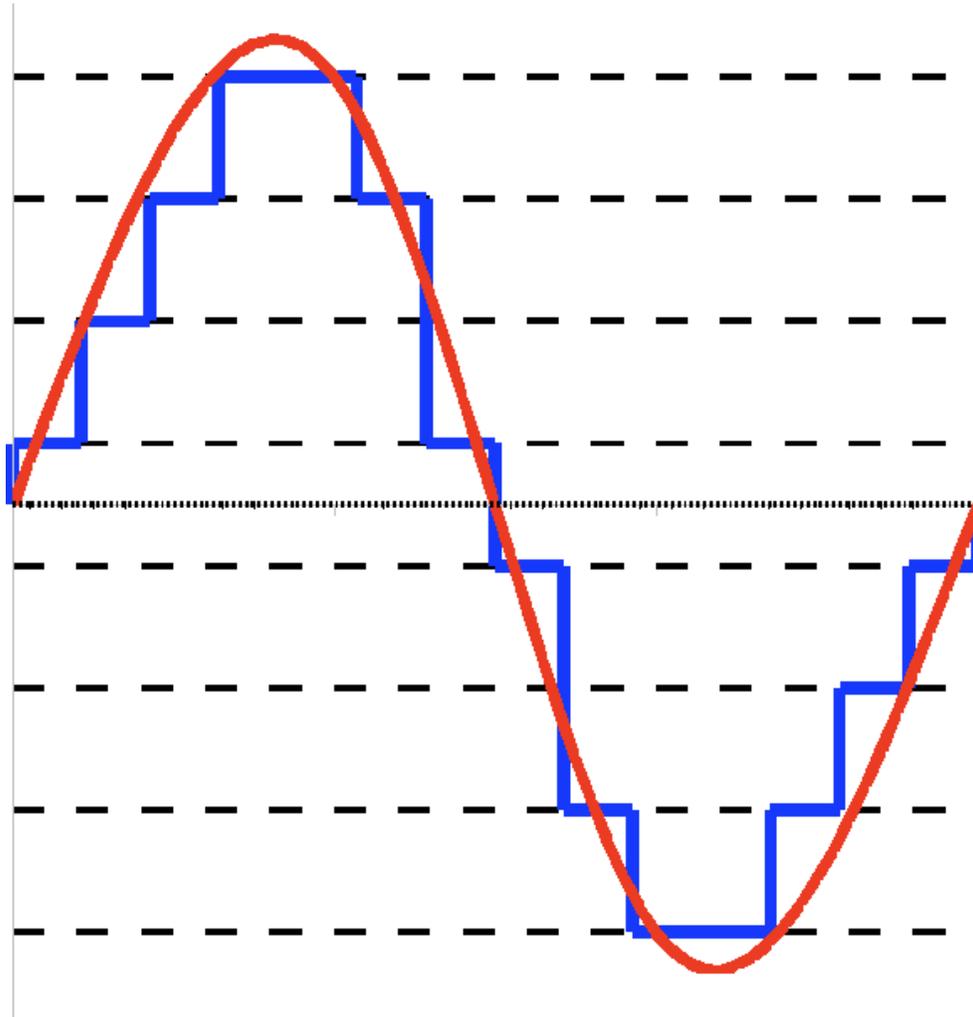
Quantization by Rounding
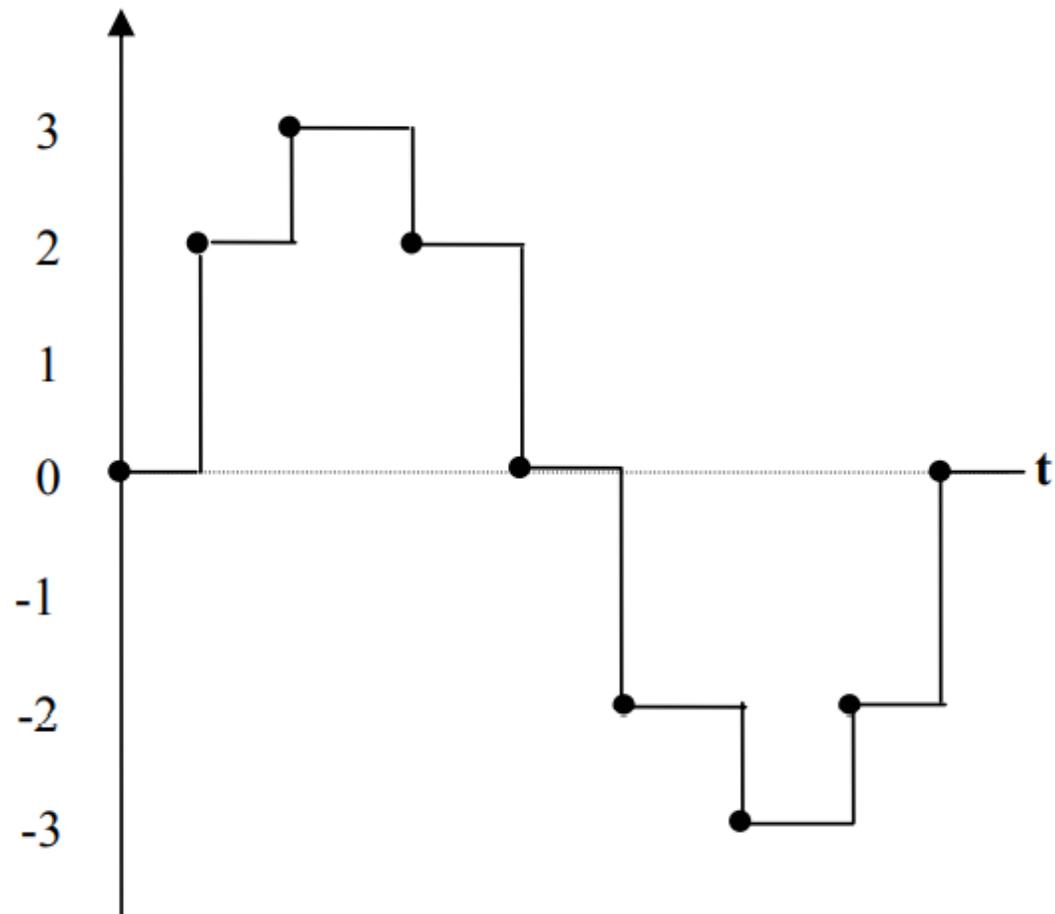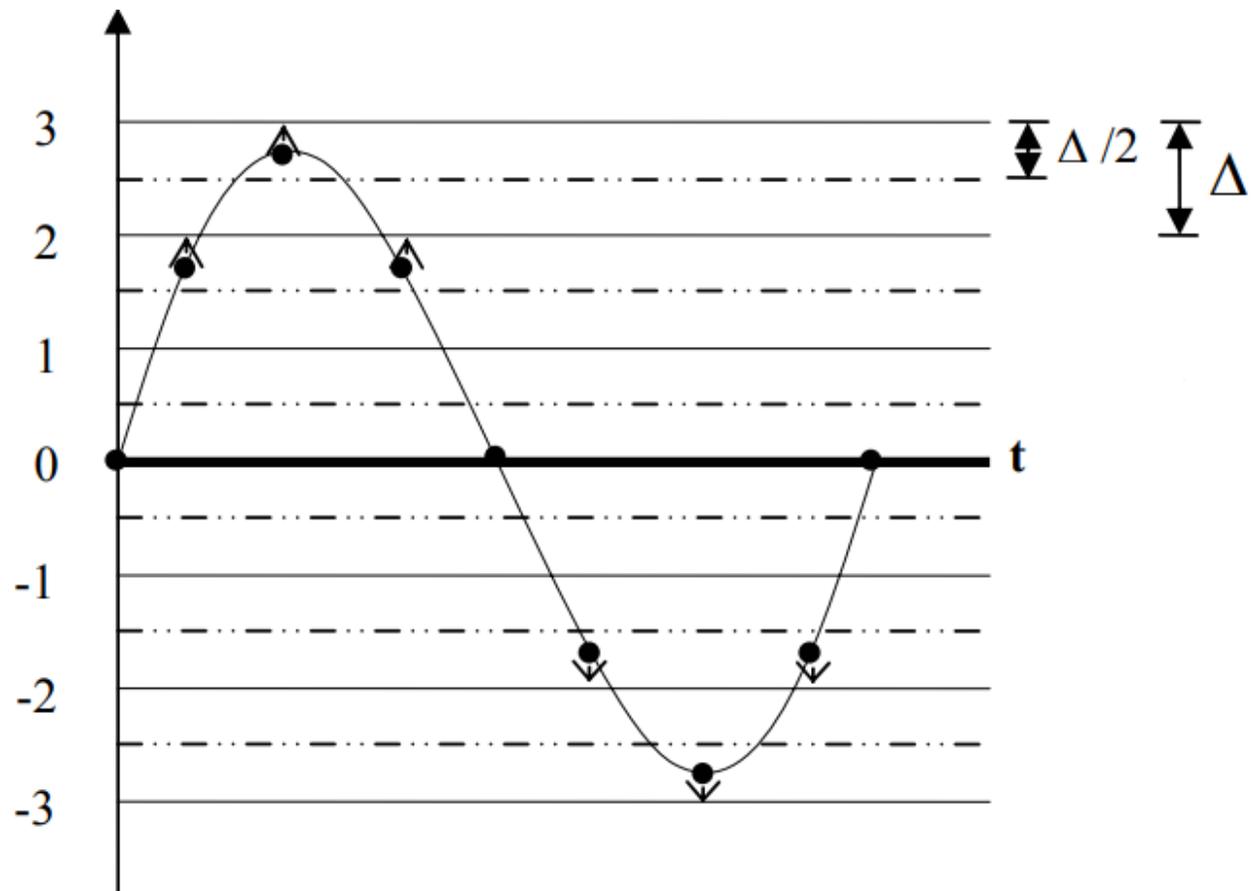
Quantization by Truncation

Question: Quantize the sequence using integer rounding and truncation: x[n]= [7.4 2.3 6.5 5.7 9.2 -0.6 -1.9 0.5 -9.6 -2 -0.5]. Also find the error sequence.

- Rounding: $x_q[n]$=[7 2 7 6 9 -1 -2 1 -10 -2 -1]
  - Error: e[n] = [-0.4 -0.3 0.5 0.3 -0.2 -0.4 -0.1 0.5 -0.4 0 -0.5]
- Truncation: $x_q[n]$ = [7 2 6 5 9 0 -1 0 -9 -2 0]
  - Error: e[n] = [-0.4 -0.3 -0.5 -0.7 -0.2 0.6 0.9 -0.5 0.6 0 0.5]

SQNR for Rounding

- x[n] = Sample input

- x_q[n] = Quantized output

- e[n] = Quantization error

- Δ = Step size or Resolution

- x_{min} = minimum of x[n]

- x_{max} = maximum of x[n]

- L = Number of level

Now, $-\dfrac{\Delta}{2} \leq e[n] \leq \dfrac{\Delta}{2}$

where

$$\Delta = \frac{x_{max} - x_{min}}{L - 1}$$
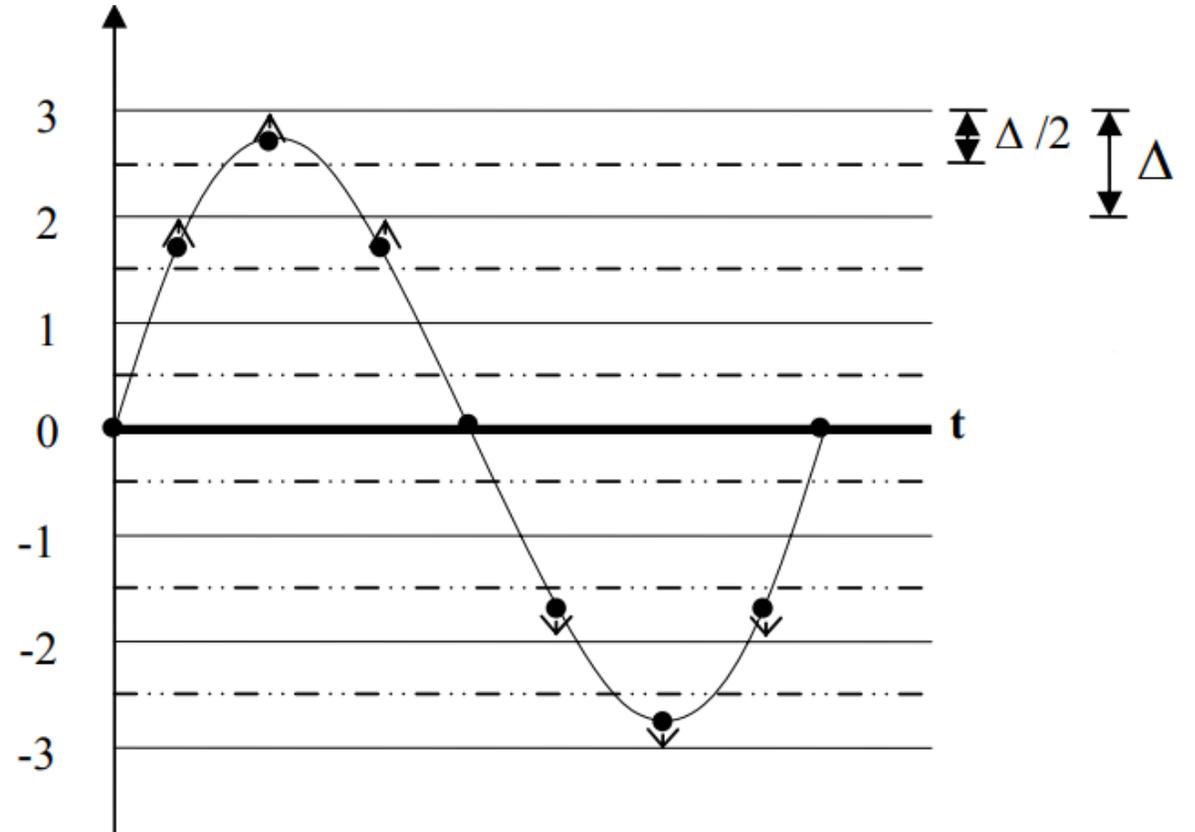
SQNR: Signal to Quantization Noise Ratio

continued...

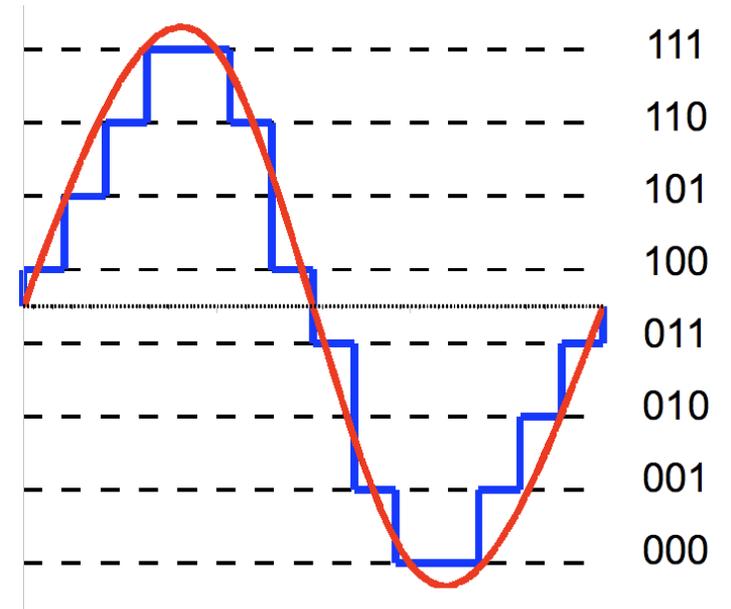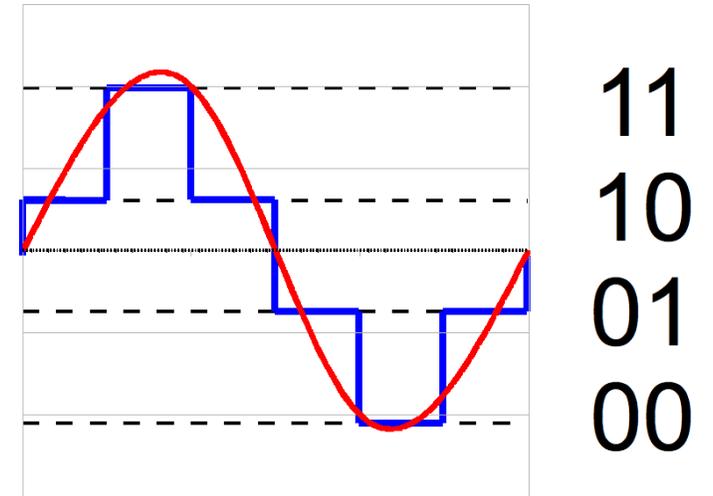If the signal is sinusoid,
then $x_{max}= A$, $x_{min}= - A$.

For coding, if number of bits used is b,

then $\Delta= \dfrac{x_{max}-x_{min}}{L-1} = \dfrac{2A}{2^b-1} \approx \dfrac{2A}{2^b}$ *[when b is higher]*

$$SQNR = \frac{P_x}{P_e} = \frac{Avg\ Power\ of\ Signal}{Avg\ Power\ of\ Error}$$

$$P_e = \frac{1}{\Delta} \int_{-\Delta/2}^{\Delta/2} e^2 de = \frac{1}{\Delta} \left[\frac{e^3}{3}\right]_{-\Delta/2}^{\Delta/2} = \frac{\Delta^2}{12}$$

And, for sinusoid, $P_x = \dfrac{A^2}{2}$

continued…

So, $SQNR = \dfrac{(A^2/2)}{(\Delta^2/12)} = \dfrac{(A^2/2)}{\left(\left(\frac{4A^2}{2^{2b}}\right)/12\right)}$ $\qquad \left[\because \Delta = \dfrac{2A}{2^b}\right]$

$$= \frac{3}{2} \cdot \left(2^{2b}\right)$$

$$\therefore SQNR(dB) = 10 \log_{10}(SQNR)$$

$$= 10 \log_{10}\left(\frac{3}{2} \cdot \left(2^{2b}\right)\right)$$

$$= 10\left(\log_{10}\left(\frac{3}{2}\right) + \log_{10}\left(2^{2b}\right)\right) \qquad \text{log(ab) = log(a) + log(b)}$$

$$= 10(0.176 + 0.6b)$$

$$= \mathbf{1.76 + 6b}$$

Question: An A/D converter uses 7-bit uniform rounding quantization for a sinusoidal signal. Find the SQNR(dB). If 8-bit was used, what would be the improvement in SQNR(dB)? If the sinusoid's amplitude is 17, find resolution in both cases.

- For 7-bit, SQNR(dB) = (1.76 + 6 x 7) dB = 43.76 dB
- For 8-bit, SQNR(dB) = (1.76 + 6 x 8) dB = 49.76 dB
    So, the improvement is (49.76 − 43.76) dB, or, 6 dB
- Resolution for 7-bit, Δ_1 = (2 x 17) / (2^7) = 0.2656
- Resolution for 8-bit, Δ_2 = (2 x 17) / (2^8) = 0.1328

Linear Constant-Coefficient Difference Equation (LCCDE)

$$y[n] + a_1 y[n-1] + a_2 y[n-2] + \cdots + a_M y[n-M]$$
$$= b_0 x[n] + b_1 x[n-1] + b_2 x[n-2] + \cdots + b_N x[n-N]$$

Solution:

- Homogeneous [zero-input], $y_h[n]$

- Particular [steady-state], $y_p[n]$

Total solution = $y[n] = y_h[n] + y_p[n]$
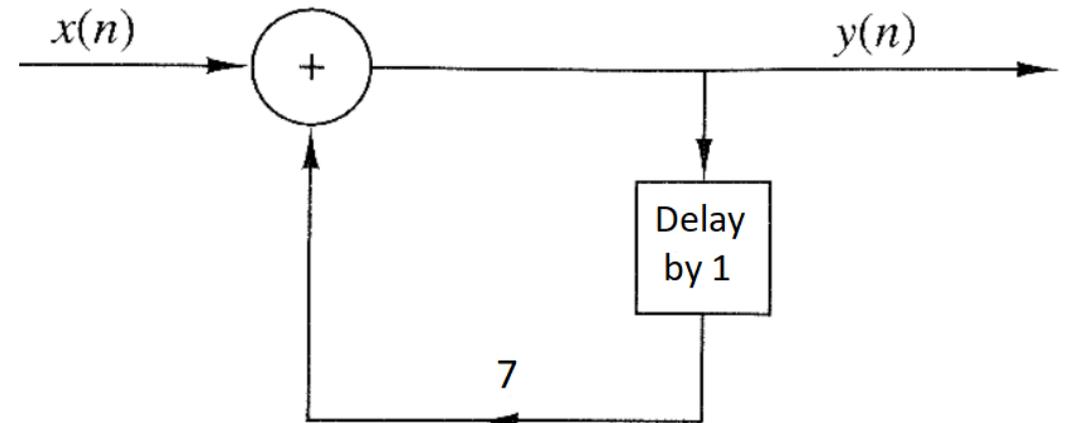
y[n] = x[n] + 7 y[n-1]
or, y[n] − 7 y[n-1] = x[n]

<u>Example:</u> Find the homogeneous solution of a system described by the 1st-order difference equation:

$$y[n] + 5y[n-1] = x[n] \qquad \text{--------(i)}$$

We know, $y_h[n] = C_1{\lambda_1}^n + C_2{\lambda_2}^n + \cdots + C_N{\lambda_N}^n$ [for order N]

Now, for eqn (i), $y_h[n] = C\lambda^n$ [because it is 1st order]

Then, we substitute an assumed solution (for x[n]=0) in eqn (i).

$$\lambda^n + 5\lambda^{n-1} = 0$$
$$\Longrightarrow \lambda^{n-1}(\lambda + 5) = 0$$
$$\Longrightarrow \lambda = -5$$

So, the solution becomes $y_h[n] = C(-5)^n$ -------(ii)

For n=0 (with x[n]=0), eqn (i) becomes $y[0] + 5y[-1] = 0$
$$\Longrightarrow y[0] = -5y[-1]$$

For n=0, eqn (ii) becomes $y_h[0] = C$

continued...

So, $C = -5y[-1]$

Then we substitute C in eqn (ii),
$$y_h[n] = (-5)y[-1](-5)^n$$
$$\implies y_h[n] = (-5)^{n+1}y[-1]$$

$$\therefore y_h[n] = (-5)^{n+1}y[-1], n \geq 0$$

Note:
If the value of y[−1] is given, it must be included in the solution.

For y[−1] = 2, $y_h[n] = 2 \cdot (-5)^{n+1}, n \geq 0$

Example: Find the homogeneous solution of a system described by the 2nd-order difference equation:

$$y[n] - 3y[n-1] - 4y[n-2] = 0 \qquad \text{-------(i)}$$

We know, $y_h[n] = C_1\lambda_1{}^n + C_2\lambda_2{}^n + \cdots + C_N\lambda_N{}^n$ [for order N]

Now, for eqn (i), $y_h[n] = C_1\lambda_1{}^n + C_2\lambda_2{}^n$ [because it is 2nd order]

Then, we substitute an assumed solution in eqn (i).

$$\lambda^n - 3\lambda^{n-1} - 4\lambda^{n-2} = 0$$
$$\Longrightarrow \lambda^{n-2}(\lambda^2 - 3\lambda - 4) = 0$$
$$\Longrightarrow \lambda^2 - 3\lambda - 4 = 0$$
$$\Longrightarrow \lambda = -1, 4$$

So, the solution becomes $y_h[n] = C_1(-1)^n + C_2(4)^n$ -----(ii)

From eqn (i), $y[n] = 3y[n-1] + 4y[n-2]$

$$\therefore y[0] = 3y[-1] + 4y[-2]$$
$$and, y[1] = 3y[0] + 4y[-1]$$

continued…

$$\Rightarrow y[1] = 3(3y[-1] + 4y[-2]) + 4y[-1]$$
$$\Rightarrow y[1] = 13y[-1] + 12y[-2]$$

From eqn (ii),
$$y[0] = C_1 + C_2$$
$$and, y[1] = -C_1 + 4C_2$$

$$\therefore C_1 + C_2 = 3y[-1] + 4y[-2] \qquad\qquad \text{-----(iii)}$$

$$and, -C_1 + 4C_2 = 13y[-1] + 12y[-2] \qquad\qquad \text{-----(iv)}$$

After solving eqn (iii) and (iv),

$$C_1 = \frac{-1}{5}y[-1] + \frac{4}{5}y[-2]$$

$$C_2 = \frac{16}{5}y[-1] + \frac{16}{5}y[-2]$$

$$\therefore y_h[n] = \left(\frac{-1}{5}y[-1] + \frac{4}{5}y[-2]\right)(-1)^n + \left(\frac{16}{5}y[-1] + \frac{16}{5}y[-2]\right)(4)^n$$

$$\text{for } n \geq 0$$

Correlation

"Measure of similarity between signals"

Use: radar, sonar, GPS etc.
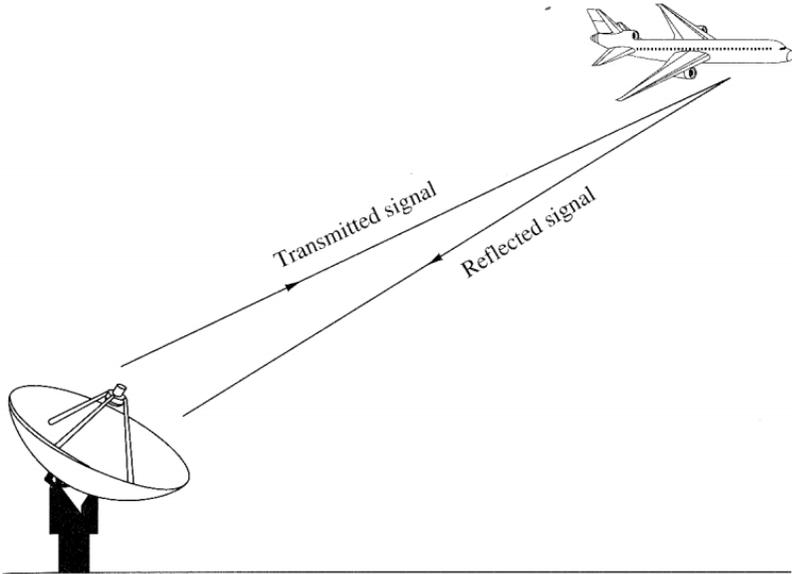
- Crosscorrelation:

$$r_{xy}(l) = \sum_{n=-\infty}^{n=\infty} x[n]\, y[n-l]$$

- Autocorrelation:

$$r_{xx}(l) = \sum_{n=-\infty}^{n=\infty} x[n]\, x[n-l]$$

Convolution vs Crosscorrelation vs Autocorrelation

Question: Find $r_{xy}[n]$ and $r_{yx}[n]$ for the following:

$x[n] = [\ 7\ \underline{4}\ 6\ 1\ ]$, $y[n] = [\ 1\ 0\ \underline{1}\ 9\ ]$.

$r_{xy}[0] = 0+4+54 = 58$

$r_{xy}[1] = 7+0+6+9 = 22$

$r_{xy}[2] = 4+0+1 = 5$

$r_{xy}[3] = 6+0 = 6$

$r_{xy}[4] = 1$

$r_{xy}[-1] = 7+36 = 43$

$r_{xy}[-2] = 63$

So, $r_{xy}[n] = [\ 63\ 43\ \underline{58}\ 22\ 5\ 6\ 1\ ]$

And, $r_{yx}[n] = [\ 1\ 6\ 5\ 22\ \underline{58}\ 43\ 63\ ]$
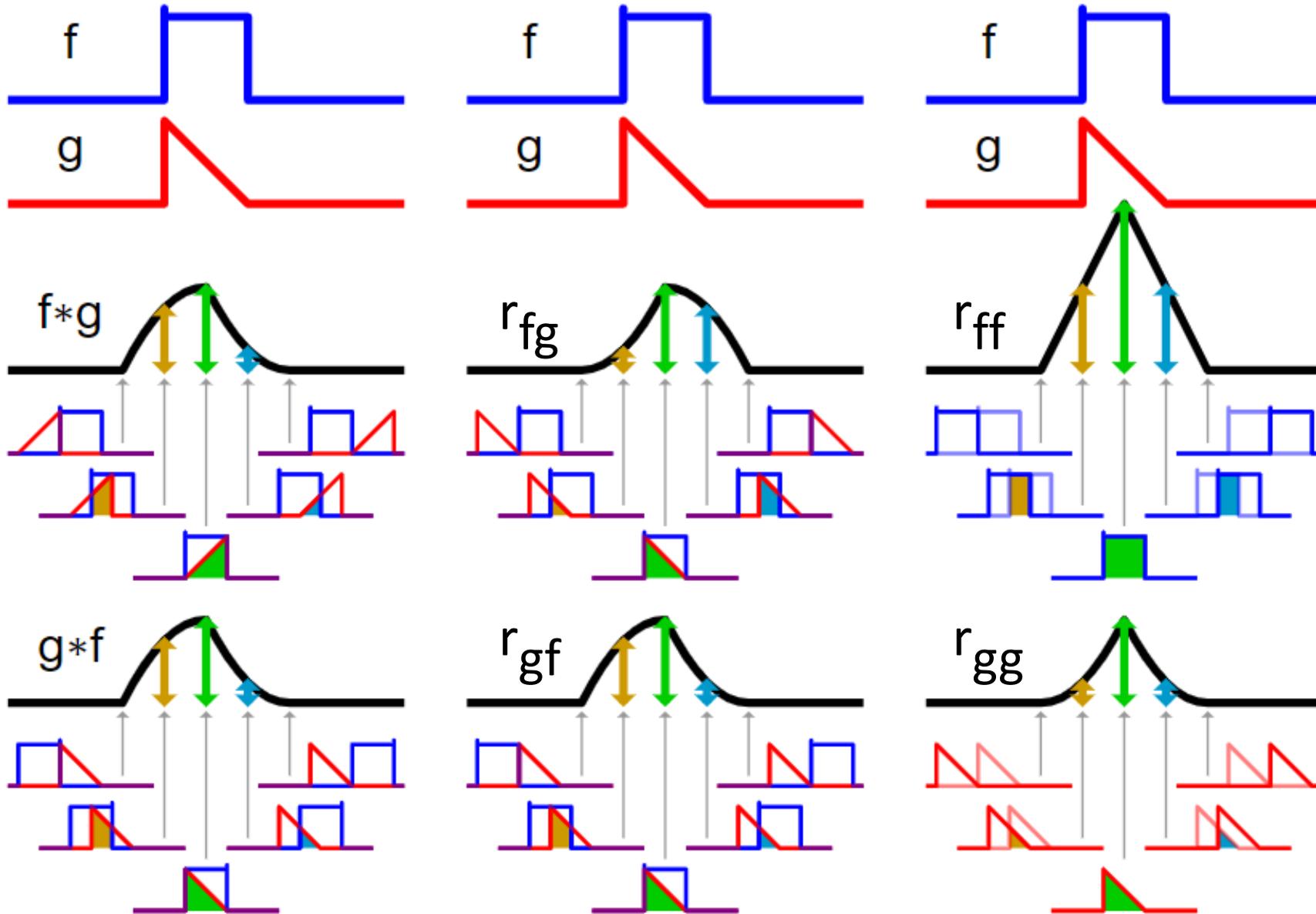
Question: Find autocorrelation for the following:

x[n]=[7 4 6]

$r_{xx}[0]$ = 49+16+36 = 101

$r_{xx}[1]$ = 28+24 = 52

$r_{xx}[2]$ = 42

$r_{xx}[-1]$ = 28+24 = 52

$r_{xx}[-2]$ = 42

So, $r_{xx}[n]$ = [42 52 101 52 42]

		7	4	6		
		7	4	6		
		7	4	6		
			7	4	6	
		7	4	6		
				7	4	6
		7	4	6		
	7	4	6			
		7	4	6		
7	4	6				

Normalized Correlation Sequence

$$\rho_{xx}[n] = \frac{r_{xx}[n]}{r_{xx}[0]}$$

$$\rho_{xy}[n] = \frac{r_{xy}[n]}{\sqrt{r_{xx}[0] \cdot r_{yy}[0]}}$$

For example, if $r_{xx}[n] = [5 \quad 10 \quad \underline{20} \quad 10 \quad 5]$

then $\rho_{xx}[n] = [0.25 \quad 0.5 \quad \underline{1} \quad 0.5 \quad 0.25]$

Discrete Time Fourier Transform (DTFT)

$$X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{n=\infty} x[n]e^{-j\omega n}$$

$X\left(e^{j\omega}\right)$ is a continuous function of ω, where ω = ΩT_s.

$X\left(e^{j\omega}\right)$ is periodic with period of 2π.

$$X\left(e^{j\omega}\right) = X\left(e^{j(\omega+2\pi k)}\right)$$ where k is any integer

To convert $X\left(e^{j\omega}\right)$ to x[n], inverse DTFT *(IDTFT)* is applied:

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X\left(e^{j\omega}\right) e^{j\omega n} d\omega$$

continued...
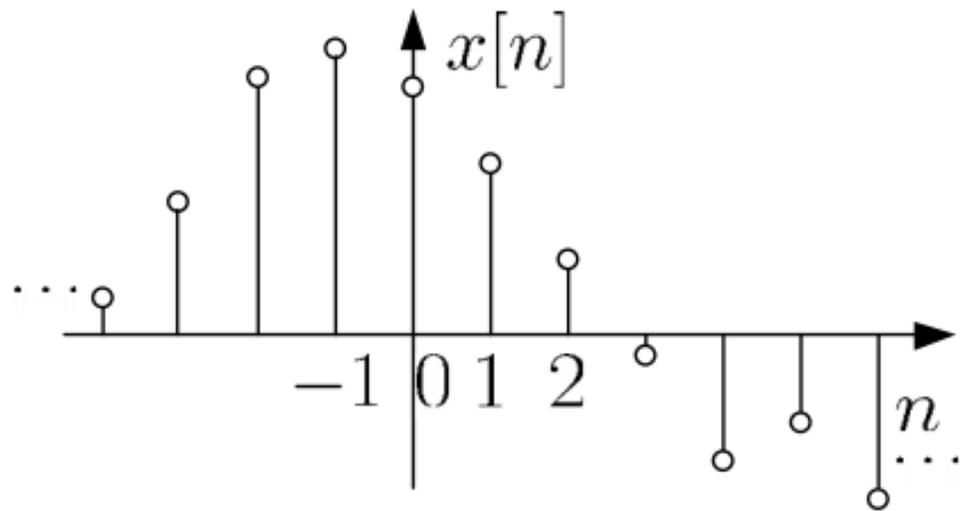
Discrete & Aperiodic

Continuous & Periodic

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X\!\left(e^{j\omega}\right) e^{j\omega n} d\omega$$

$$X\!\left(e^{j\omega}\right) = \sum_{n=-\infty}^{n=\infty} x[n] e^{-j\omega n}$$

Existence of DTFT

- Sufficient condition:

$$\left|X\left(e^{j\omega}\right)\right| \leq \sum_{n=-\infty}^{n=\infty} |x[n]| < \infty$$

x[n] must be absolutely summable.

Then, DTFT *converges* to a finite result for all **ω**.

Spectra

$X\left(e^{j\omega}\right)$ is generally complex.

- Magnitude spectrum:
$$\left|X\left(e^{j\omega}\right)\right|$$
$$= \sqrt{(\Re\{X(e^{j\omega})\})^2 + (\Im\{X(e^{j\omega})\})^2}$$

- Phase spectrum:
$$\sphericalangle\left(X\left(e^{j\omega}\right)\right) = \tan^{-1}\left(\frac{\Im\left(X(e^{j\omega})\right)}{\Re\left(X(e^{j\omega})\right)}\right)$$

Note: Both are continuous & periodic.

Unit Impulse

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$
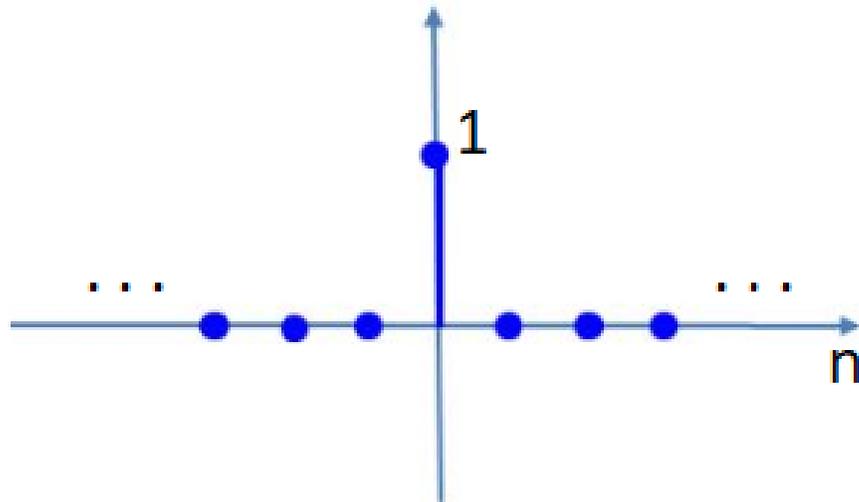
$\delta[n] = [\dots 0\ 0\ 0\ 0\ \underline{1}\ 0\ 0\ 0\ 0 \dots]$

Unit Step

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$
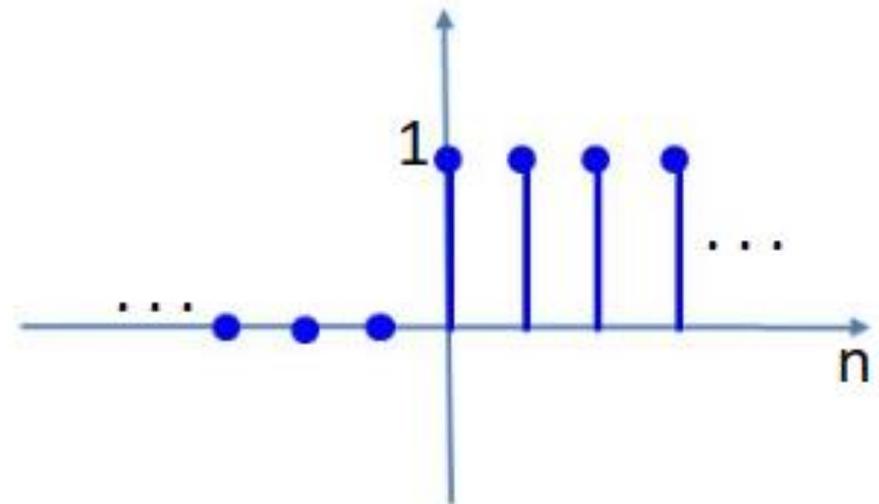
$u[n] = [\dots 0\ 0\ 0\ 0\ \underline{1}\ 1\ 1\ 1\ 1 \dots]$

Question: Find the DTFT of x[n] = u[n] − u[n-3]

u[n]   = [… 0 0 0 0 0 1 1 1 1 1 1 1 1 …]

u[n-3]   = [… 0 0 0 0 0 0 0 0 1 1 1 1 1 …]

x[n]   = [… 0 0 0 0 0 1 1 1 0 0 0 0 0 …]

We know,

$$X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{n=\infty} x[n]e^{-j\omega n}$$

$$= \sum_{n=0}^{n=2} x[n]e^{-j\omega n}$$

$$= x[0]e^{-j\omega 0} + x[1]e^{-j\omega 1} + x[2]e^{-j\omega 2}$$

$$= 1 + e^{-j\omega} + e^{-2j\omega}$$

Question: Find the DTFT of $x[n] = (1/2)^n u[n]$

We know,

$$X(e^{j\omega}) = \sum_{n=-\infty}^{n=\infty} x[n] e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{n=\infty} (1/2)^n u[n] e^{-j\omega n}$$

But, u[n] = 0 when n < 0.

So,

$$X(e^{j\omega}) = \sum_{n=0}^{n=\infty} (1/2)^n e^{-j\omega n}$$

u[n]
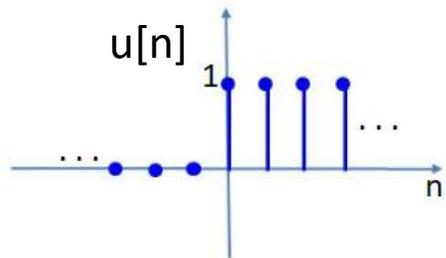
$$\Rightarrow X(e^{j\omega}) = \sum_{n=0}^{n=\infty} \left(\frac{1}{2} e^{-j\omega}\right)^n$$

$$= \left(\frac{1}{2} e^{-j\omega}\right)^0 + \left(\frac{1}{2} e^{-j\omega}\right)^1 + \left(\frac{1}{2} e^{-j\omega}\right)^2 + \cdots$$

$$= 1 + \left(\frac{1}{2} e^{-j\omega}\right)^1 + \left(\frac{1}{2} e^{-j\omega}\right)^2 + \cdots$$

This is a convergent geometric series (because the |ratio| is < 1)

$$\therefore X(e^{j\omega}) = \frac{1}{1 - \frac{1}{2} e^{-j\omega}}$$

[using formula $S_\infty = \frac{a}{1-r}$]

Question: Find the IDTFT of

$$X(e^{j\omega}) = \begin{cases} 1, & -\pi/4 \leq \omega \leq \pi/4 \\ 0, & otherwise \end{cases}$$
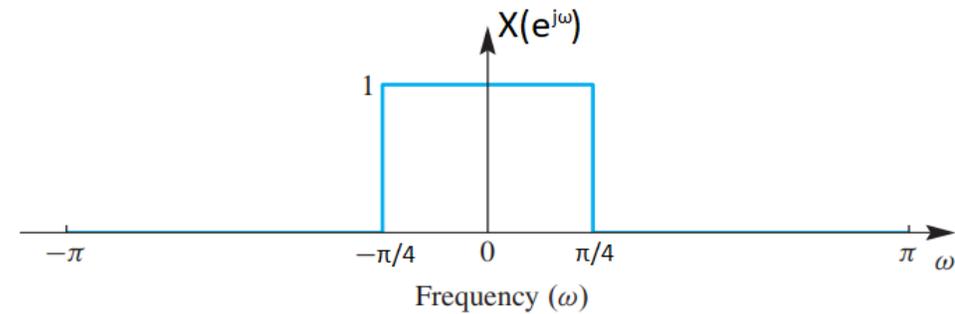
We know,

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

$$\Longrightarrow x[n] = \frac{1}{2\pi} \int_{-\pi/4}^{\pi/4} e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi} \cdot \frac{1}{jn} \left[e^{j\omega n} \right]_{-\pi/4}^{\pi/4}$$

$$= \frac{1}{\pi n} \cdot \frac{1}{2j} \left(e^{\left(j\frac{\pi}{4}n \right)} - e^{\left(-j\frac{\pi}{4}n \right)} \right)$$

$$= \frac{1}{\pi n} \sin\left(\frac{\pi}{4} n \right)$$

[We know, $\sin\theta = \dfrac{e^{j\theta} - e^{-j\theta}}{2j}$]

$$\Longrightarrow x[n] = \frac{1}{4} \cdot \frac{1}{\frac{\pi}{4}n} \sin\left(\frac{\pi}{4} n \right)$$

$$\Longrightarrow x[n] = \frac{1}{4} sinc\left(\frac{\pi}{4} n \right)$$

$$x[n] = \frac{1}{4} \, sinc \left(\frac{\pi}{4} n\right)$$
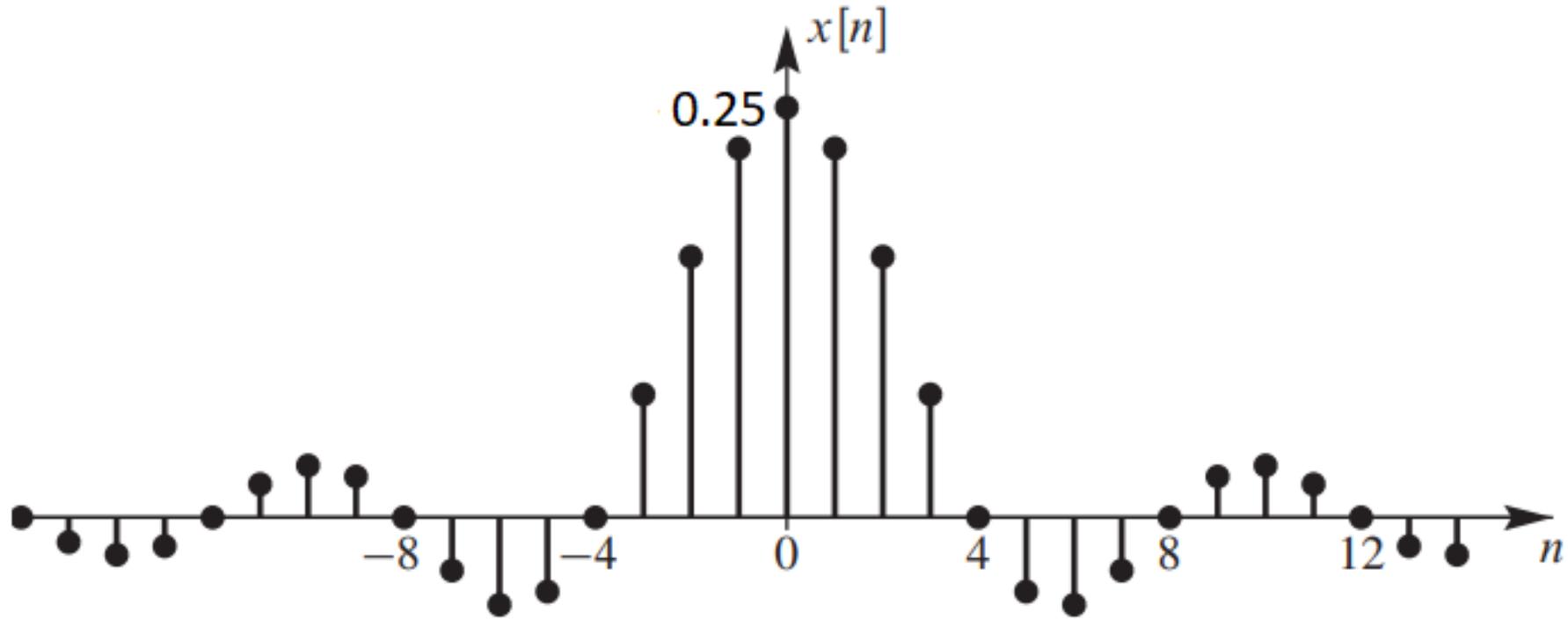
Some Properties of DTFT

- Linearity

$$x[n] = ax_1[n] + bx_2[n] \overset{DTFT}{\longleftrightarrow} X(e^{j\omega}) = aX_1(e^{j\omega}) + bX_2(e^{j\omega})$$

- Time Delay

$$y[n] = x[n-k] \overset{DTFT}{\longleftrightarrow} Y(e^{j\omega}) = X(e^{j\omega})e^{-j\omega k}$$

- Frequency Shift

$$y[n] = e^{j\omega_c n}x[n] \overset{DTFT}{\longleftrightarrow} Y(e^{j\omega}) = X(e^{j(\omega-\omega_c)})$$

- Convolution

$$y[n] = x[n] * h[n] \overset{DTFT}{\longleftrightarrow} Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$

- Differentiation

$$y[n] = nx[n] \overset{DTFT}{\longleftrightarrow} Y(e^{j\omega}) = j\frac{d}{d\omega}\left(X(e^{j\omega})\right)$$

continued…

- Time Reversal

$$y[n] = x[-n] \overset{DTFT}{\longleftrightarrow} Y(e^{j\omega}) = X(e^{-j\omega})$$

- Conjugation

$$y[n] = x^*[n] \overset{DTFT}{\longleftrightarrow} Y(e^{j\omega}) = X^*(e^{-j\omega})$$

- Modulation

$$y[n] = x[n]\cos(\omega_0 n) \overset{DTFT}{\longleftrightarrow} Y(e^{j\omega}) = \frac{1}{2}X(e^{j(\omega-\omega_0)}) + \frac{1}{2}X(e^{j(\omega+\omega_0)})$$

- Multiplication

$$x[n] = x_1[n] \cdot x_2[n] \overset{DTFT}{\longleftrightarrow} X(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(e^{j\tau})X_2(e^{j(\omega-\tau)})\, d\tau$$

- Parseval's Theorem

$$\sum_{n=-\infty}^{n=\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2\, d\omega$$

Discrete Fourier Transform (DFT)

Recall the DTFT of x[n],

$$X(\omega) = \sum_{n=-\infty}^{n=\infty} x[n]e^{-j\omega n}$$

Let's sample X(ω) periodically in frequency at a spacing of $\delta\omega$.

Since X(ω) is periodic with period 2π, only samples in the fundamental frequency range are necessary.
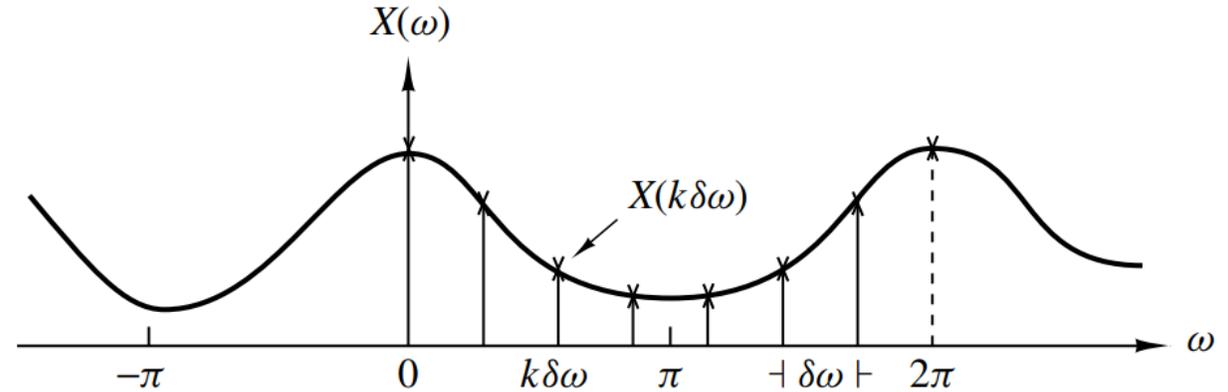
We take N equidistant samples in the interval $0 \leq \omega < 2\pi$ with spacing $\delta\omega = \frac{2\pi}{N}$

$$X\left(\frac{2\pi}{N}k\right) = \sum_{n=-\infty}^{n=\infty} x[n]e^{-j2\pi kn/N}$$

where k = 0,1,2,...,N−1

continued…

But, the equally spaced frequency samples $X\left(\frac{2\pi}{N}k\right)$ do not uniquely represent the original sequence x[n] when x[n] has infinite duration.

Instead, the frequency samples correspond to a periodic sequence of period N, that is an aliased version of x[n].

So, a finite-duration sequence x[n] of length L has a DTFT:

$$X(\omega) = \sum_{n=0}^{n=L-1} x[n]e^{-j\omega n} \; ; 0 \leq \omega \leq 2\pi$$

After equidistant sampling of X(ω) for N ≥ L,

$$X(k) \equiv X\left(\frac{2\pi}{N}k\right) = \sum_{n=0}^{n=L-1} x[n]e^{-j2\pi kn/N}$$

$$\Longrightarrow X(k) = \sum_{n=0}^{n=N-1} x[n]e^{-j2\pi kn/N}$$

where k = 0,1,2,…,N−1

The N-point inverse DFT *(IDFT)*:

$$x[n] = \frac{1}{N} \sum_{k=0}^{k=N-1} X(k)e^{j2\pi kn/N}$$

where n = 0,1,2,…,N−1

continued...

Let's see the DTFT and DFT of $x[n] = \begin{cases} 1, & 0 \leq n \leq L - 1 \\ 0, & otherwise \end{cases}; L = 10$
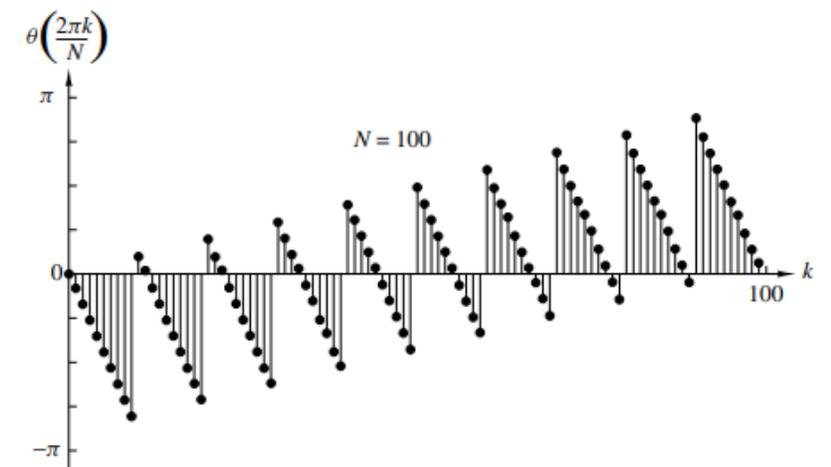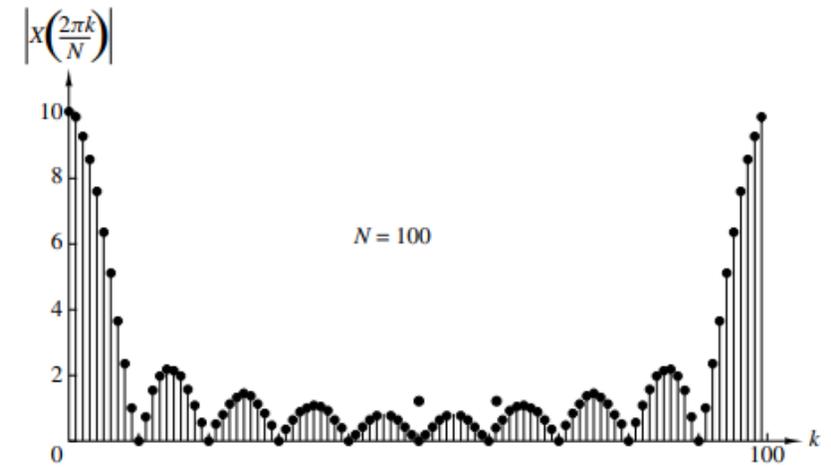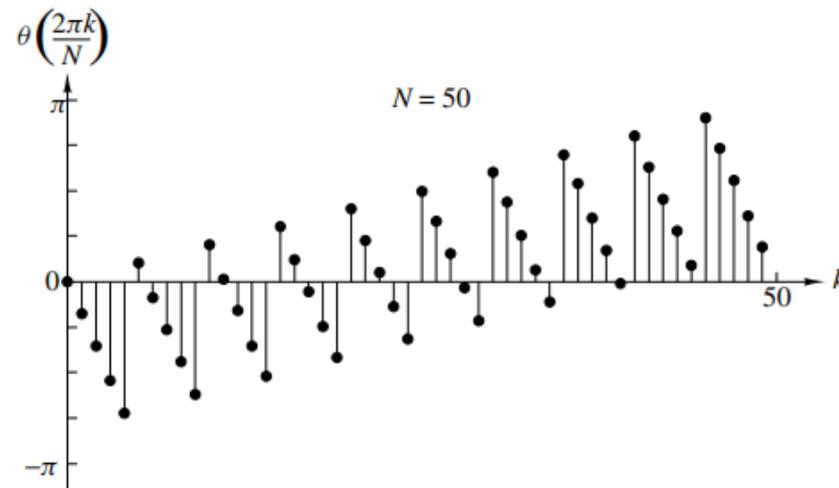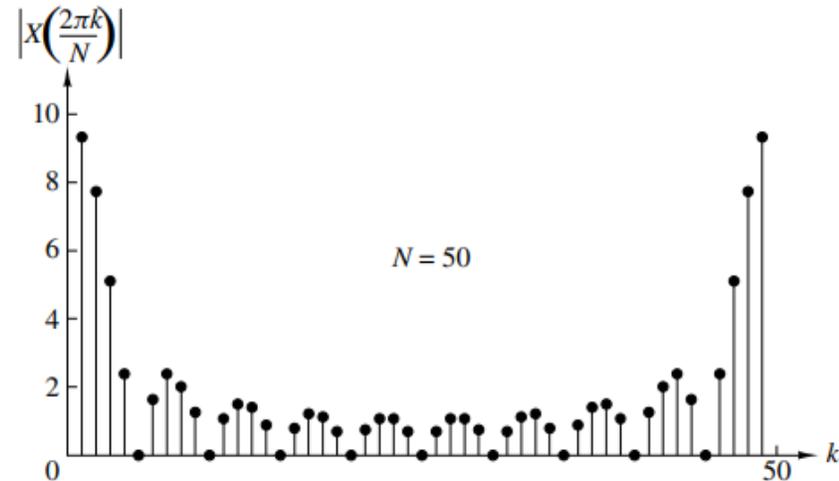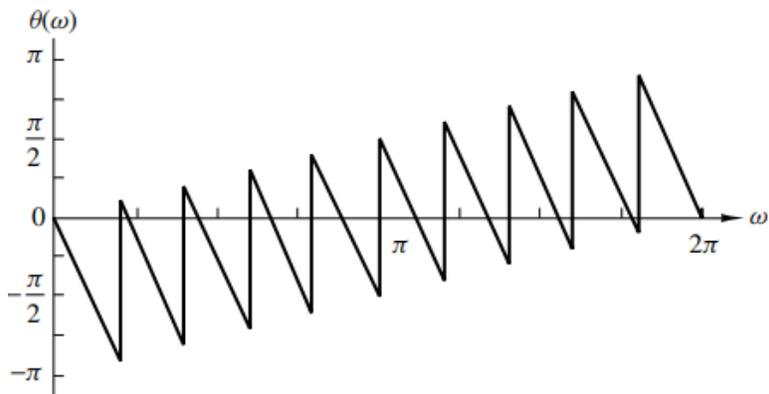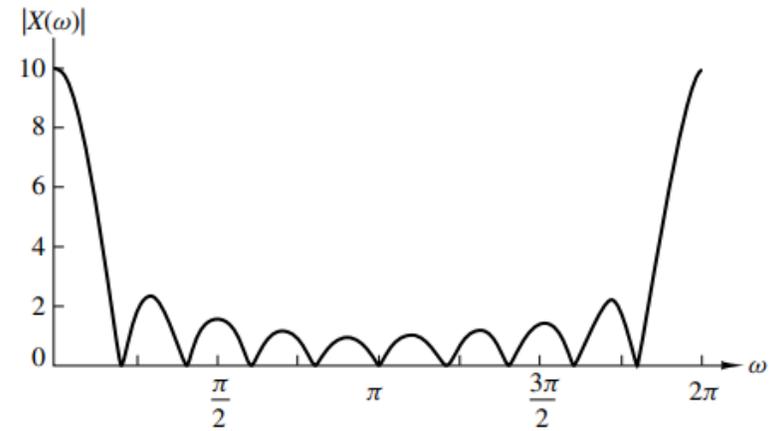
Example: Find the 4-point DFT of x[n]=[2 1 7 5]

$$X(k) = \sum_{n=0}^{n=3} x[n]e^{-j2\pi nk/4}$$

where k = 0, 1, 2, 3

$$\Rightarrow X(k) = x[0]e^0 + x[1]e^{-j2\pi k/4} + x[2]e^{-j2\pi 2k/4} + x[3]e^{-j2\pi 3k/4}$$
$$\Rightarrow X(k) = 2 + e^{-j\pi k/2} + 7e^{-j\pi k} + 5e^{-j3\pi k/2}$$

- $X(0) = 2 + 1 + 7 + 5 = 15$

- $X(1) = 2 + e^{-j\pi/2} + 7e^{-j\pi} + 5e^{-j3\pi/2} = -5 + 4j$

- $X(2) = 2 + e^{-j\pi} + 7e^{-j\pi 2} + 5e^{-j3\pi} = 3$

- $X(3) = 2 + e^{-j\pi 3/2} + 7e^{-j\pi 3} + 5e^{-j3\pi 3/2} = -5 - 4j$

Magnitude, $|X(k)| = \begin{bmatrix} 15 & 6.403 & 3 & 6.403 \end{bmatrix}$

Angle, $\angle X(k) = \begin{bmatrix} 0 & 2.467 & 0 & -2.467 \end{bmatrix}$

z-Transform

$$X(z) = \sum_{n=-\infty}^{n=\infty} x[n]z^{-n}$$

where z is a *complex* variable.

$$x[n] \overset{z}{\leftrightarrow} X(z)$$

The Region of Convergence (ROC) of X(z) is the set of values of z for which X(z) attains a finite value.

Example: Determine the z-transforms of the following finite-duration signals:

- $x_1[n]=[\ 9\ \ 2\ \ \underline{4}\ \ 6\ \ 3\ \ 1\ ]$

  $X_1(z)=9z^2+2z+4+6z^{-1}+3z^{-2}+z^{-3}$,
  ROC: entire z-plane except z=0 and z=∞

- $x_2[n]=[\ \underline{1}\ \ 0\ \ 7\ \ 9\ \ 5\ ]$

  $X_2(z)=1+7z^{-2}+9z^{-3}+5z^{-4}$,
  ROC: entire z-plane except z=0

More example: Determine the z-transforms of the following finite-duration signals:

- $x_3[n] = \delta[n]$
  $X_3(z) = 1$;   ROC: entire z-plane

- $x_4[n] = \delta[n-6]$
  $X_4(z) = z^{-6}$;   ROC: entire z-plane except $z = 0$

- $x_5[n] = \delta[n+7]$
  $X_5(z) = z^7$;   ROC: entire z-plane except $z = \infty$

- $x_6[n] = [\ 4\ \ 6\ \ 0\ \ \underline{7}\ ]$

  $X_6(z) = 4z^3 + 6z^2 + 7$; ROC: entire z-plane except $z = \infty$

Example: Determine the z-transform of the signal:

$$x[n] = \left(\frac{1}{3}\right)^n u[n]$$

We get, $x[n] = \left[\begin{array}{ccccc} \underline{1} & \frac{1}{3} & \left(\frac{1}{3}\right)^2 & \left(\frac{1}{3}\right)^3 & \left(\frac{1}{3}\right)^4 \cdots \end{array}\right]$

Thus, $X(z) = 1 + \frac{1}{3}z^{-1} + \left(\frac{1}{3}\right)^2 z^{-2} + \left(\frac{1}{3}\right)^3 z^{-3} + \left(\frac{1}{3}\right)^4 z^{-4} + \cdots$

$$= 1 + \left(\frac{1}{3}z^{-1}\right) + \left(\frac{1}{3}z^{-1}\right)^2 + \left(\frac{1}{3}z^{-1}\right)^3 + \left(\frac{1}{3}z^{-1}\right)^4 + \cdots$$

Recall for infinite geometric series, $S_\infty = \dfrac{a}{1-r}$, where $|r| < 1$

So, $\left|\frac{1}{3}z^{-1}\right| < 1 \Longrightarrow |z| > \frac{1}{3}$

$$\therefore X(z) = \frac{1}{1 - \frac{1}{3}z^{-1}} ; ROC: |z| > \frac{1}{3}$$

Existence of X(z)

Let, $z = re^{j\theta}$, where $r = |z|$, and $\theta = \sphericalangle z$

$$X(z) = \sum_{n=-\infty}^{n=\infty} x[n]z^{-n} = \sum_{n=-\infty}^{n=\infty} x[n]r^{-n}e^{-j\theta n}$$

In the ROC of X(z), $|X(z)| < \infty$

$$|X(z)| = \left| \sum_{n=-\infty}^{n=\infty} x[n]r^{-n}e^{-j\theta n} \right| \leq \sum_{n=-\infty}^{n=\infty} |x[n]r^{-n}|$$

So, $|X(z)|$ is finite if $x[n]r^{-n}$ is absolutely summable.

$$|X(z)| \leq \sum_{n=-\infty}^{n=-1} |x[n]r^{-n}| + \sum_{n=0}^{n=\infty} |x[n]r^{-n}|$$

continued...

$$|X(z)| \leq \sum_{n=1}^{n=\infty} |x[-n]r^n| + \sum_{n=0}^{n=\infty} |x[n]r^{-n}|$$

- ROC for 1st sum consists of all points inside a circle of radius r_1

- ROC for 2nd sum consists of all points outside a circle of radius r_2

Thus, X(z) exists if $r_2 < r < r_1$

❑ If $r_2 > r_1$, there is no common ROC, and X(z) doesn't exist.

(a) Region of convergence for $\sum_{n=1}^{\infty} |x(-n) r^n|$

(b) Region of convergence for $\sum_{n=0}^{\infty} \left|\frac{x(n)}{r^n}\right|$

(c) Region of convergence for $|X(z)|$
$r_2 < r < r_1$
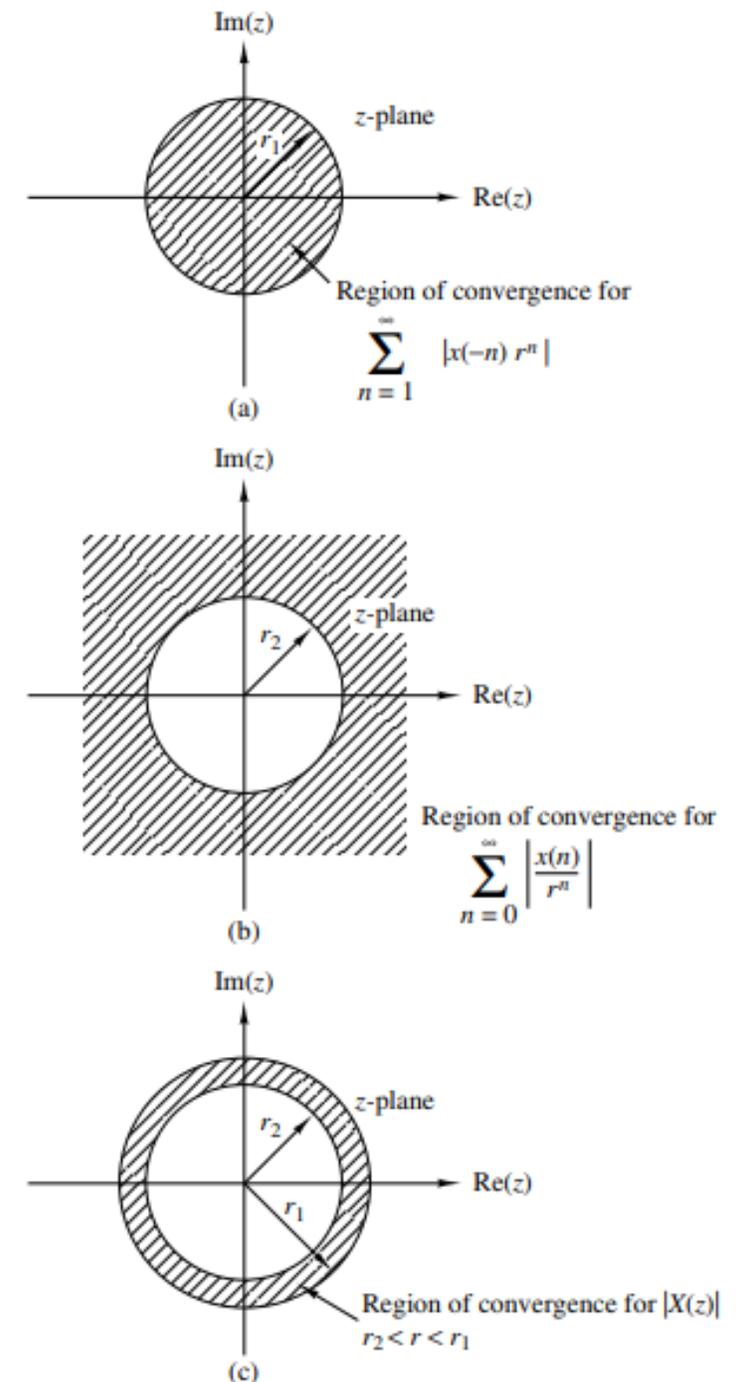
continued...

Recall the example:

$$x[n] = \left(\frac{1}{3}\right)^n u[n] \overset{z}{\leftrightarrow} X(z) = \frac{1}{1 - \frac{1}{3}z^{-1}}; ROC: |z| > \frac{1}{3}$$

Similarly,

$$x[n] = (a)^n u[n] \overset{z}{\leftrightarrow} X(z) = \frac{1}{1 - az^{-1}}; ROC: |z| > |a|$$

where a can be real or complex.
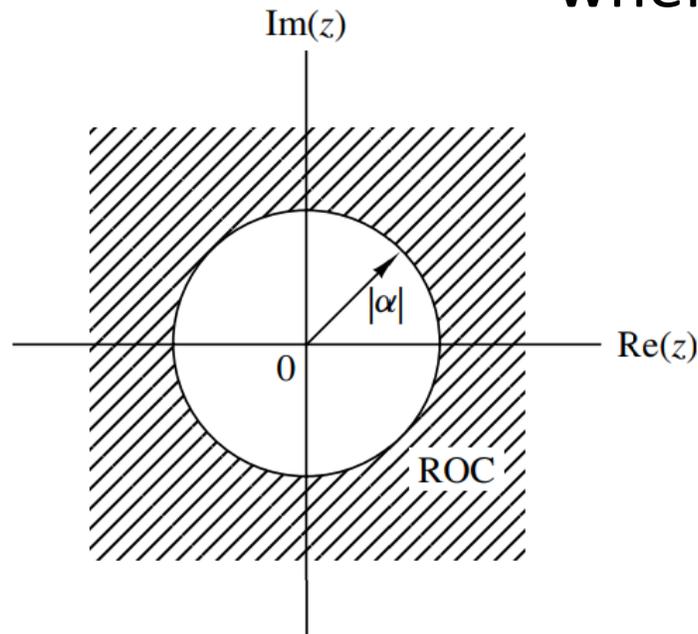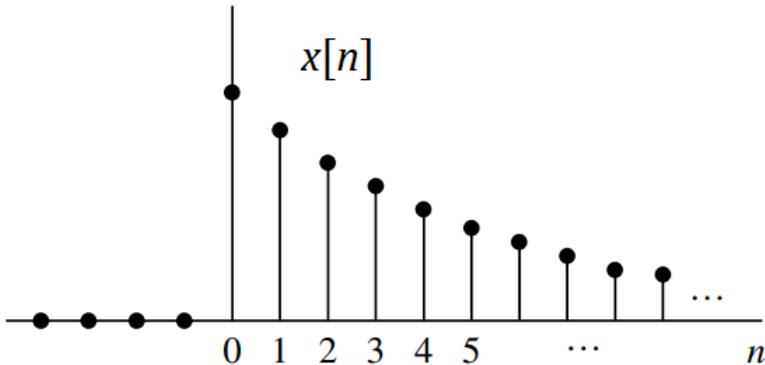
Example: Determine the z-transform of the signal:
$$x[n] = -5^n u[-n-1]$$

We get, $x[n] = [\ldots \quad -5^{-4} \quad -5^{-3} \quad -5^{-2} \quad -5^{-1}] = [\ldots \quad \frac{-1}{5^4} \quad \frac{-1}{5^3} \quad \frac{-1}{5^2} \quad \frac{-1}{5}]$

Thus, $X(z) = \cdots + \left(\frac{-1}{5^4}\right) z^4 + \left(\frac{-1}{5^3}\right) z^3 + \left(\frac{-1}{5^2}\right) z^2 + \left(\frac{-1}{5}\right) z^1$

$$= -\left(\ldots + \left(\frac{1}{5}z\right)^4 + \left(\frac{1}{5}z\right)^3 + \left(\frac{1}{5}z\right)^2 + \left(\frac{1}{5}z\right)^1\right)$$

Recall for infinite geometric series, $S_\infty = \dfrac{a}{1-r}$, where $|r| < 1$

So, $\left|\frac{1}{5}z\right| < 1 \implies |z| < 5$

$u[-n-1]$

$$\therefore X(z) = -\frac{\frac{1}{5}z}{1 - \frac{1}{5}z} = \frac{1}{1 - 5z^{-1}} ; ROC : |z| < 5$$
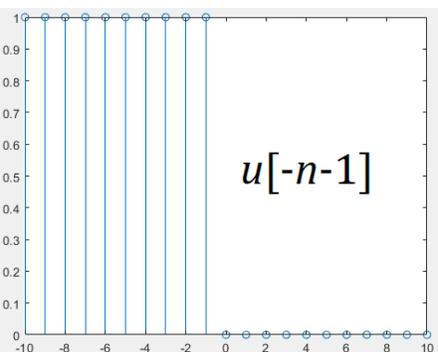
continued...

Recall the previous example:

$$x[n] = -5^n u[-n-1] \overset{z}{\leftrightarrow} X(z) = \frac{1}{1 - 5z^{-1}}; ROC: |z| < 5$$

Similarly,

$$x[n] = -a^n u[-n-1] \overset{z}{\leftrightarrow} X(z) = \frac{1}{1 - az^{-1}}; ROC: |z| < |a|$$

where a can be real or complex.
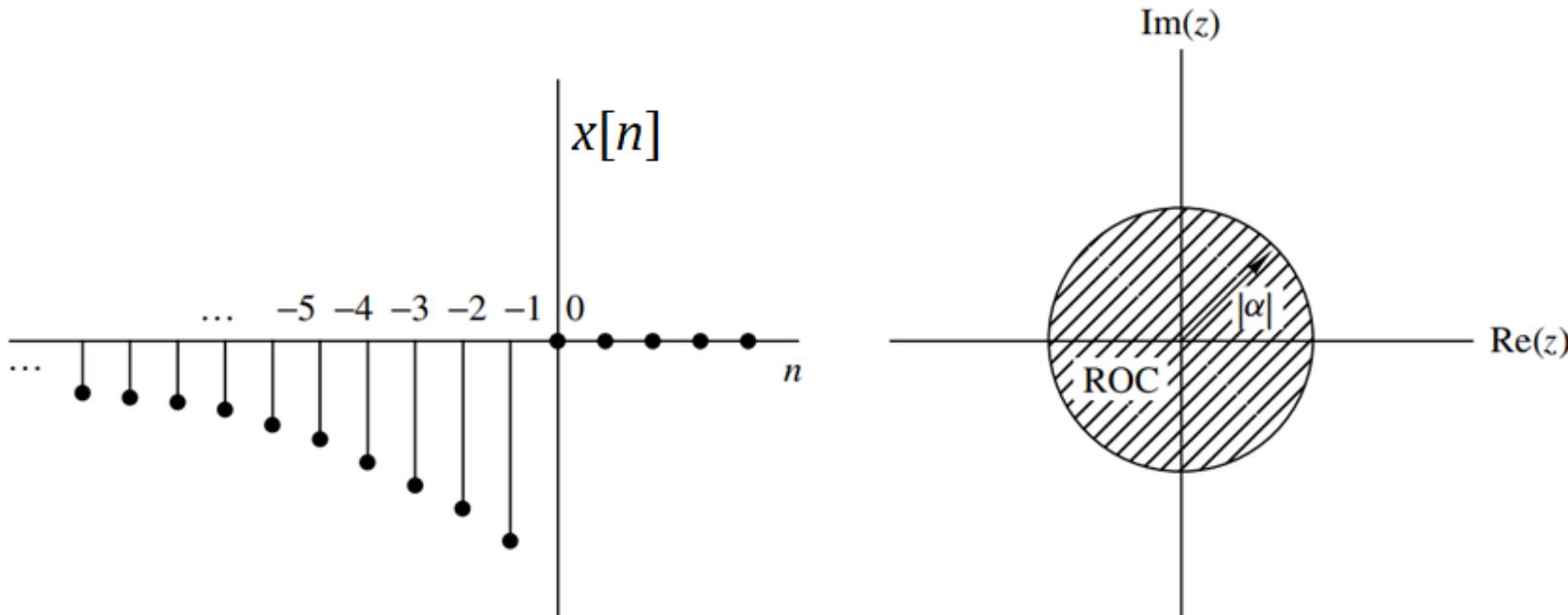
Example: Determine the z-transform of the signal:
$$x[n] = a^n u[n] + b^n u[-n-1]$$

For the 1st term, $ROC: |z| > |a|$

For the 2nd term, $ROC: |z| < |b|$

- If $|a| > |b|$, two ROC don't overlap. So, X(z) doesn't exist.

- If $|a| < |b|$, the ROC is ring-shaped. X(z) exists. $ROC: |a| < |z| < |b|$

For 2nd case,
$$X(z) = \frac{1}{1 - az^{-1}} - \frac{1}{1 - bz^{-1}}$$

Im(z)

z-plane

$|b|$

$|\alpha|$

Re(z)

$|b| < |\alpha|$
X(z) does not exist

Im(z)

$|\alpha| < |b|$

z-plane

$|b|$

$|\alpha|$

Re(z)

ROC for X(z)
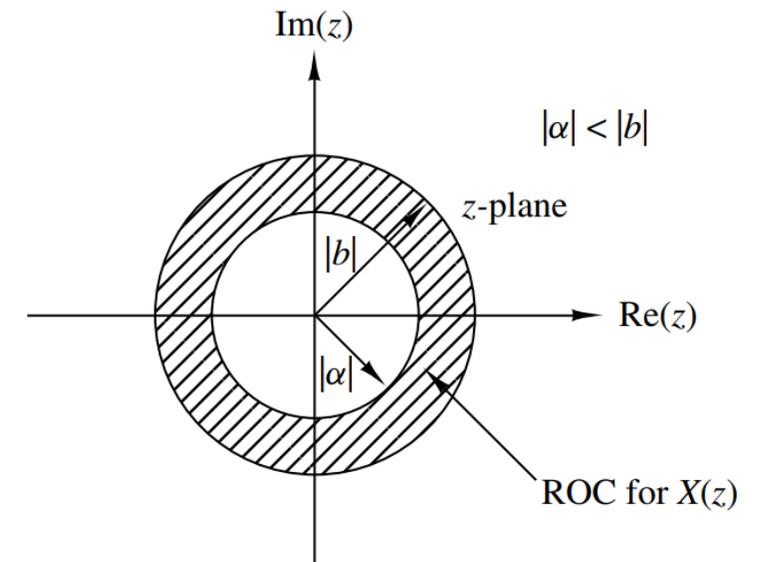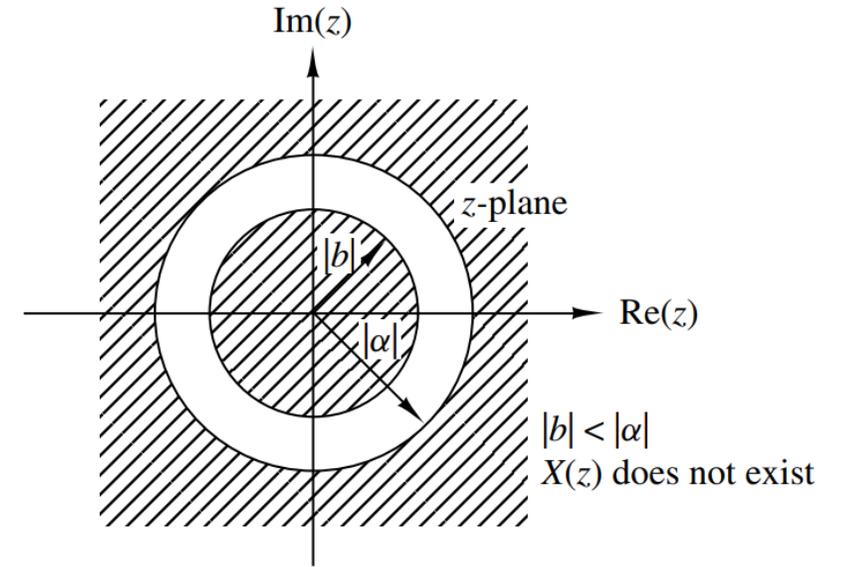
ROC Characteristic

Finite-Duration Signal   **Infinite-Duration Signal**
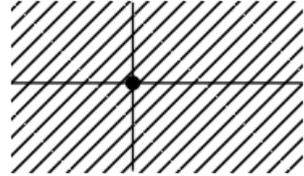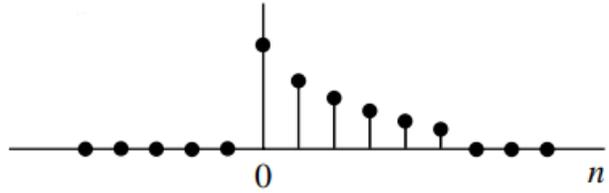
Entire z-plane except $z = 0$
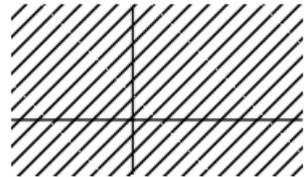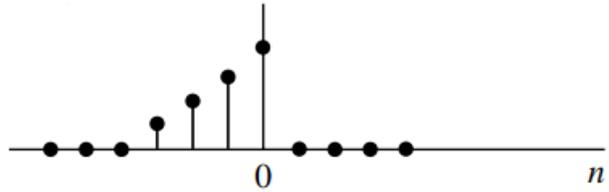
Entire z-plane except $z = \infty$
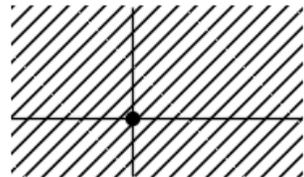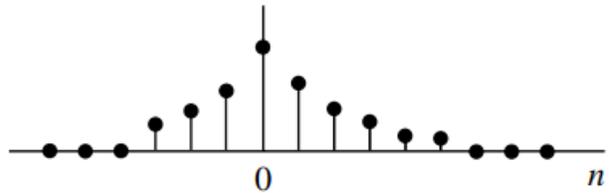
Entire z-plane except $z = 0$ and $z = \infty$
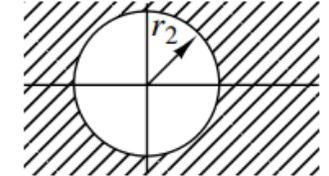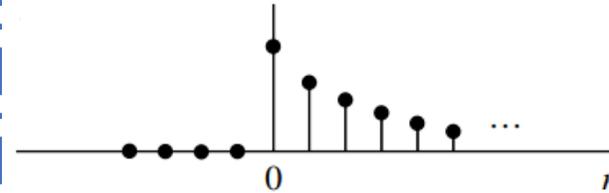
$|z| > r_2$
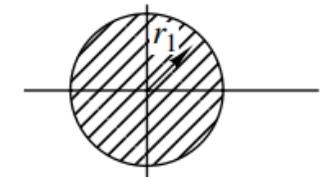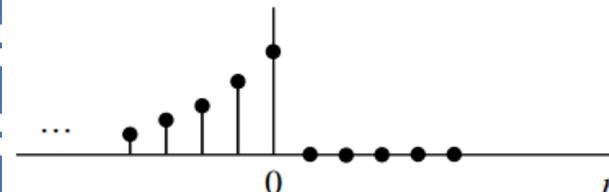
$|z| < r_1$
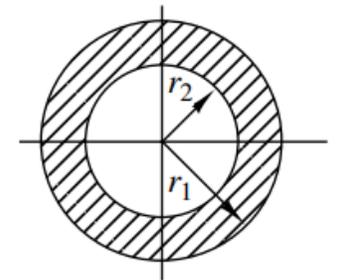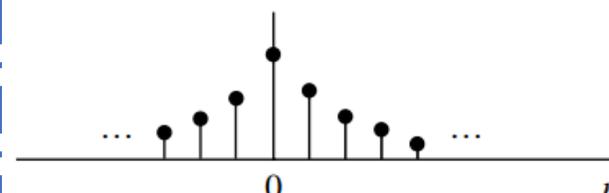
$r_2 < |z| < r_1$

(Causal)
(Anti-Causal)
(Two-Sided)

Some Properties of z-Transform

Linearity: $x[n] = ax_1[n] + bx_2[n] \overset{z}{\leftrightarrow} X(z) = aX_1(z) + bX_2(z)$

Example: Find the z-transform for $x[n] = \left(10(4^n) - 17(9^n)\right)u[n]$

We get, $x[n] = 10(4^n)u[n] - 17(9^n)u[n]$
$$= 10x_1[n] - 17x_2[n]$$

- $x_1[n] = 4^n u[n] \overset{z}{\leftrightarrow} X_1(z) = \dfrac{1}{1-4z^{-1}}; ROC: |z| > 4$

- $x_2[n] = 9^n u[n] \overset{z}{\leftrightarrow} X_2(z) = \dfrac{1}{1-9z^{-1}}; ROC: |z| > 9$

$$\therefore X(z) = \frac{10}{1 - 4z^{-1}} - \frac{17}{1 - 9z^{-1}}; ROC: |z| > 9$$

<u>Time-Shifting:</u> $y[n] = x[n-k] \overset{z}{\leftrightarrow} Y(z) = z^{-k}X(z)$

Example: Using time-shifting, find z-transform for $y_1[n]=x[n+3]$ and $y_2[n]=x[n-2]$; where x[n]=[<u style="color:red">2</u> 0 5 4 6 3].

We get, $X(z) = 2 + 5z^{-2} + 4z^{-3} + 6z^{-4} + 3z^{-5}; ROC: |z| > 0$

- $Y_1(z) = z^3 X(z)$
$$= 2z^3 + 5z + 4 + 6z^{-1} + 3z^{-2}; ROC: 0 < |z| < \infty$$

- $Y_2(z) = z^{-2}X(z)$
$$= 2z^{-2} + 5z^{-4} + 4z^{-5} + 6z^{-6} + 3z^{-7}; ROC: |z| > 0$$

continued...

Scaling: If $x[n] \overset{z}{\leftrightarrow} X(z); ROC: r_1 < |z| < r_2$

Then $a^n x[n] \overset{z}{\leftrightarrow} X(a^{-1}z); ROC: |a|r_1 < |z| < |a|r_2$

Time Reversal: If $x[n] \overset{z}{\leftrightarrow} X(z); ROC: r_1 < |z| < r_2$

Then $x[-n] \overset{z}{\leftrightarrow} X(z^{-1}); ROC: \dfrac{1}{r_2} < |z| < \dfrac{1}{r_1}$

Example: Determine the z-transform for x[n]=u[-n].

Recall that, $u[n] \overset{z}{\leftrightarrow} \dfrac{1}{1-z^{-1}}; ROC: |z| > 1$

$$\therefore u[-n] \overset{z}{\leftrightarrow} \dfrac{1}{1-z}; ROC: |z| < 1$$

continued...

Differentiation: $y[n] = nx[n] \overset{z}{\leftrightarrow} Y(z) = -z\frac{dX(z)}{dz}; ROC\ unchanged$

Example: Determine the z-transform for $y[n] = n(6^n)u[n]$

We recall,

$$x[n] = a^n u[n] \overset{z}{\leftrightarrow} X(z) = \frac{1}{1 - az^{-1}}; ROC: |z| > |a|$$

$$x[n] = 6^n u[n] \overset{z}{\leftrightarrow} X(z) = \frac{1}{1 - 6z^{-1}}; ROC: |z| > |6|$$

$$\therefore y[n] = nx[n] \overset{z}{\leftrightarrow} Y(z) = -z\frac{dX(z)}{dz}; ROC: |z| > 6$$

$$\implies Y(z) = -z\frac{d}{dz}\left(\frac{1}{1 - 6z^{-1}}\right) = \frac{6z^{-1}}{(1 - 6z^{-1})^2}; ROC: |z| > 6$$

Convolution: $x[n] = x_1[n] * x_2[n] \overset{z}{\leftrightarrow} X(z) = X_1(z)X_2(z)$

Example: Using convolution property of z-transform, find $x[n] = x_1[n] * x_2[n]$, where x$_1$=[<u>5</u> 0 7], x$_2$=[<u>2</u> 4]

- $X_1(z) = 5 + 7z^{-2}; ROC: |z| > 0$
- $X_2(z) = 2 + 4z^{-1}; ROC: |z| > 0$

$$\therefore X(z) = X_1(z)X_2(z)$$
$$\Longrightarrow X(z) = (5 + 7z^{-2})(2 + 4z^{-1})$$
$$= 10 + 20z^{-1} + 14z^{-2} + 28z^{-3}; ROC: |z| > 0$$
$$\therefore x[n] = [\ \underline{10}\quad 20\quad 14\quad 28\ ]$$

Poles & Zeros for Rational X(z)

$$X(z) = z^{N-M} \cdot \frac{b_0(z - z_1)(z - z_2) \dots (z - z_M)}{a_0(z - p_1)(z - p_2) \dots (z - p_N)}$$

❑ The zeros of X(z) are the values of z for which $X(z) = 0$

❑ The poles of X(z) are the values of z for which $X(z) = \infty$

Note:

• ROC of X(z) should not contain any poles.

• If N > M, then X(z) has (N–M) zeros at origin.

• If M > N, then X(z) has (M–N) poles at origin.

• Poles or zeros may also occur at $z = \infty$

Example: Determine the pole-zero plot for the signal:
$$x[n] = 4^n u[n]$$

We recall,

$$X(z) = \frac{1}{1 - 4z^{-1}}; ROC: |z| > 4$$

$$\Rightarrow X(z) = \frac{z}{z - 4}; ROC: |z| > 4$$

X(z) has one zero at z=0

X(z) has one pole at z=4

Time-Domain Behavior with Pole Location
"Causal Sequence $x[n] = a^n u[n]$"
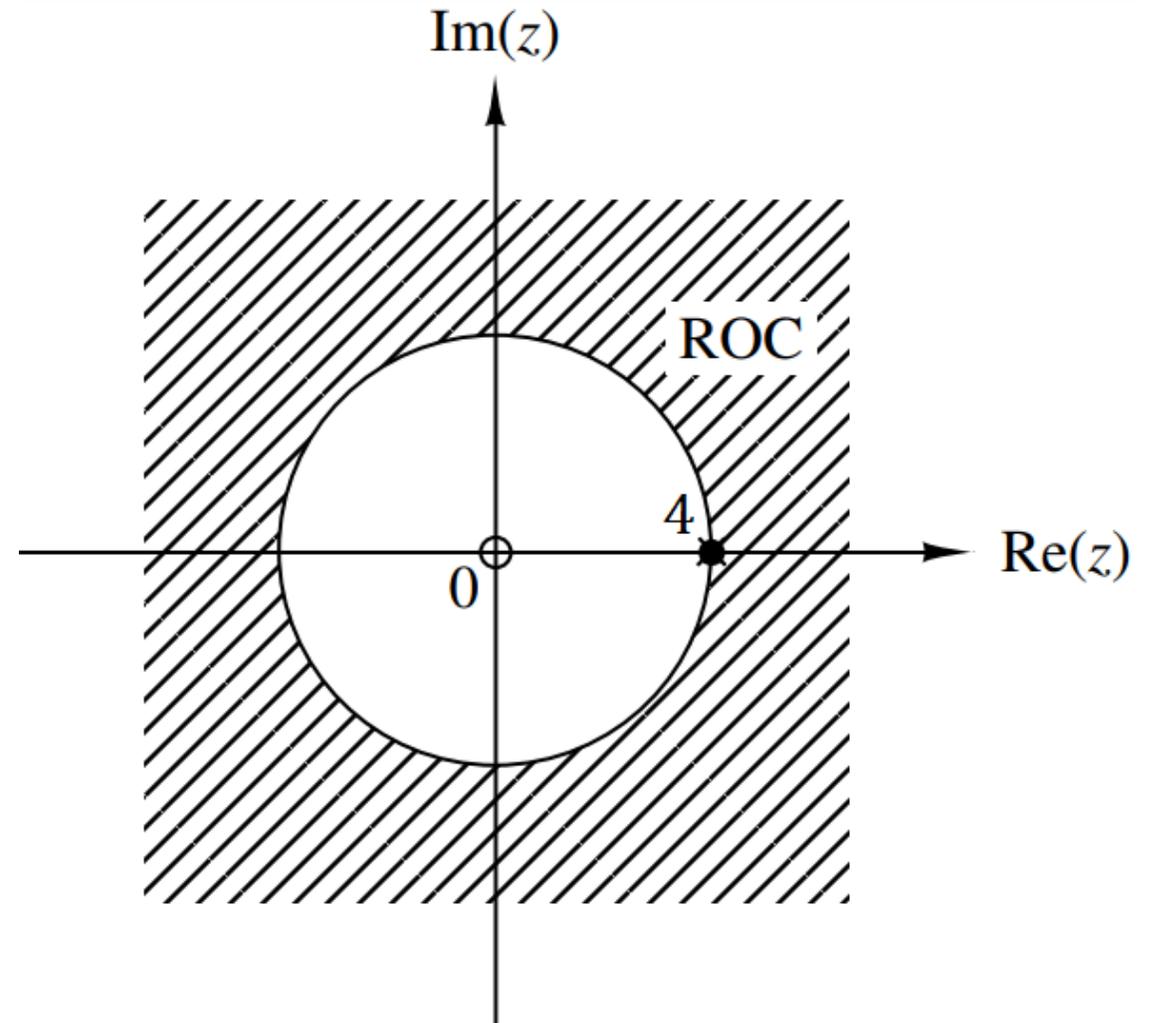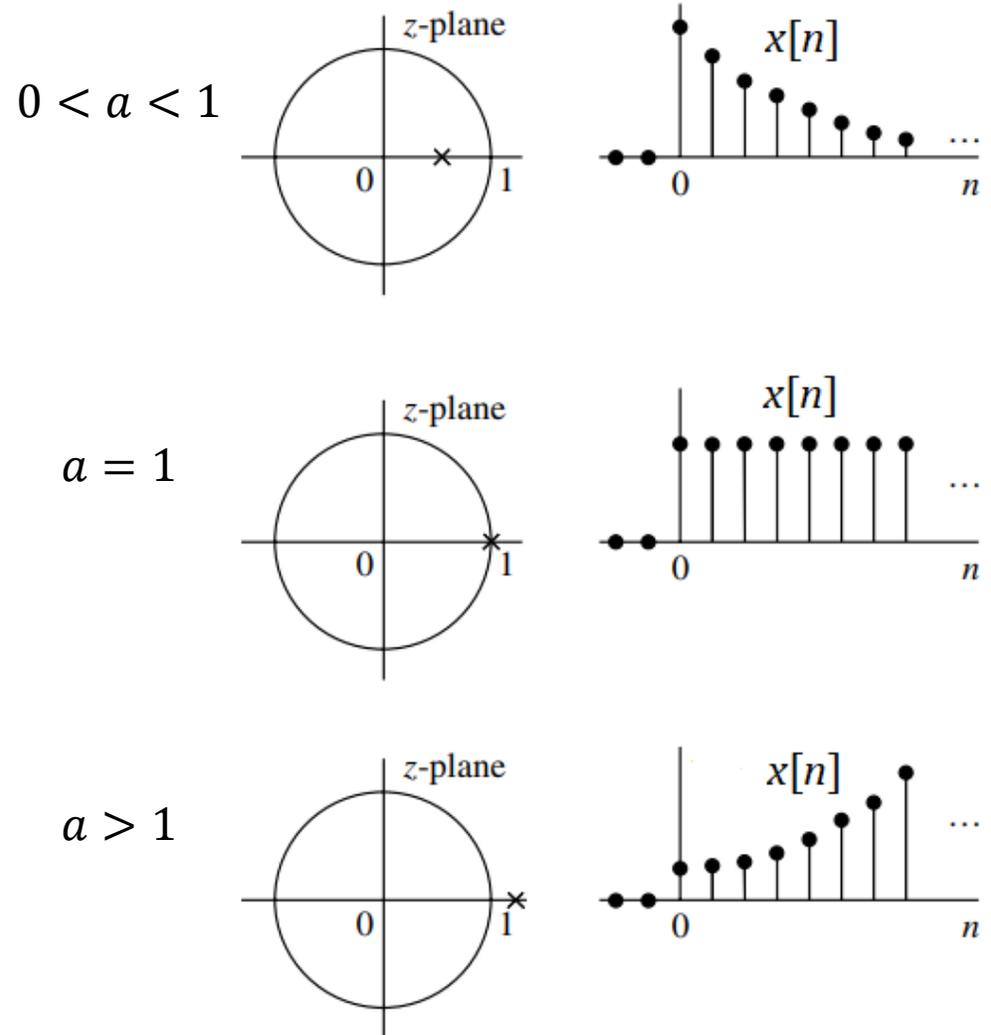
Time-Domain Behavior with Pole Location
"Causal Sequence $x[n] = na^n u[n]$ "

Double Real Pole (Positive)
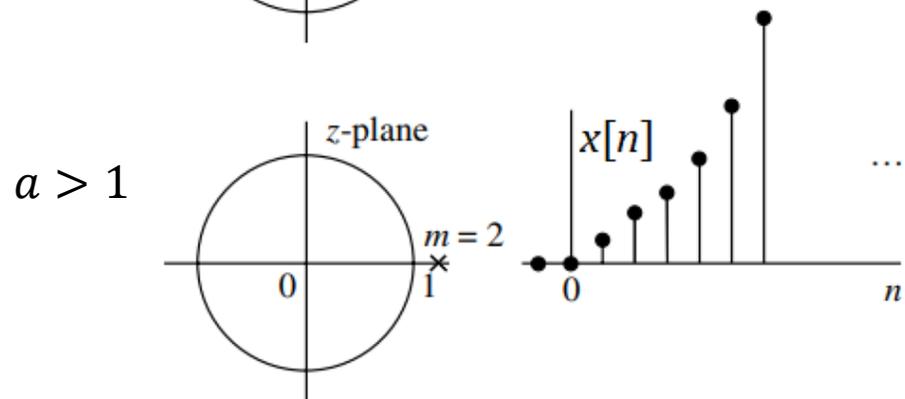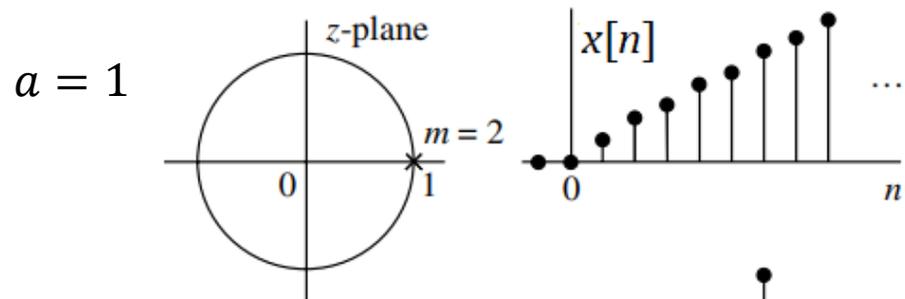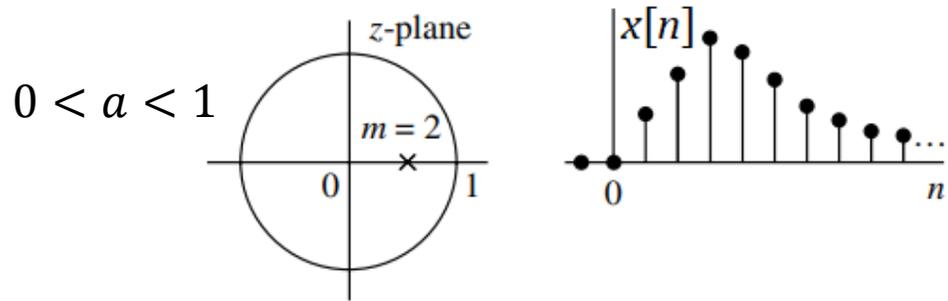
Double Real Pole (Negative)

$0 < a < 1$

$a = 1$

$a > 1$

$-1 < a < 0$

$a = -1$

$a < -1$
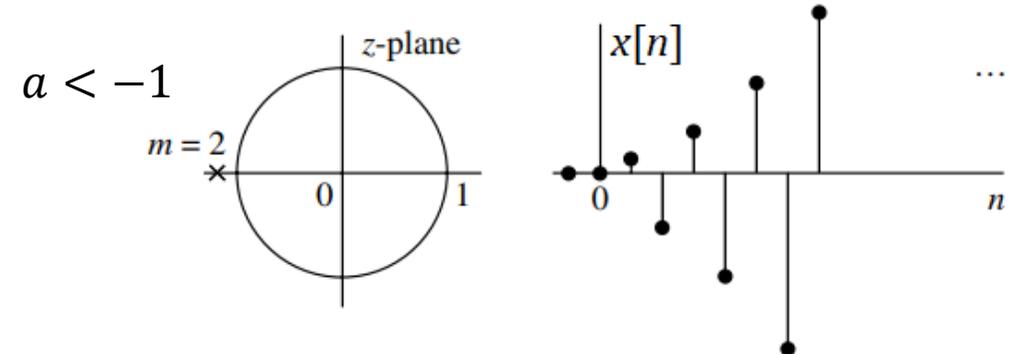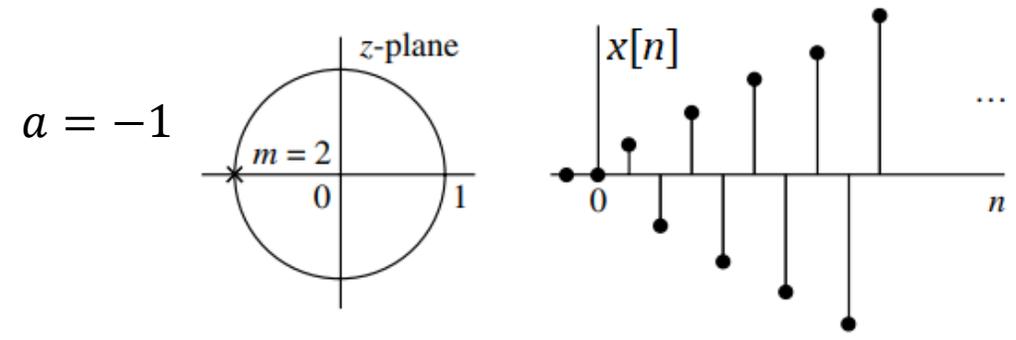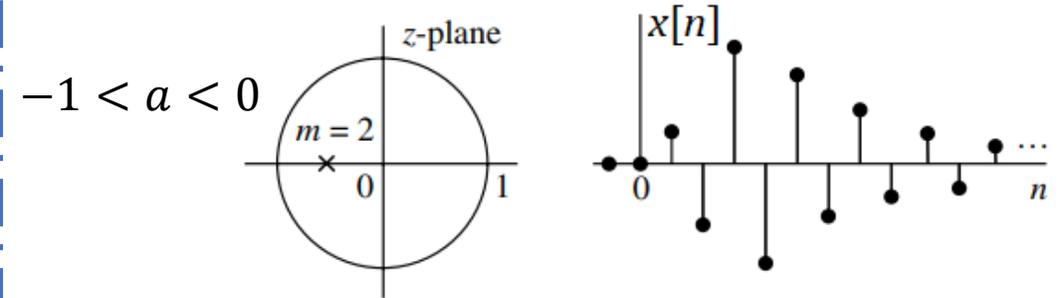
continued...

- Causal real signals with simple real poles or simple complex conjugate pairs of poles, which are inside or on the unit circle, are always bounded in amplitude.

- Time behavior of a signal depends strongly on the location of its poles relative to the unit circle.

- Zeros also affect the behavior of a signal but not as strongly as poles.
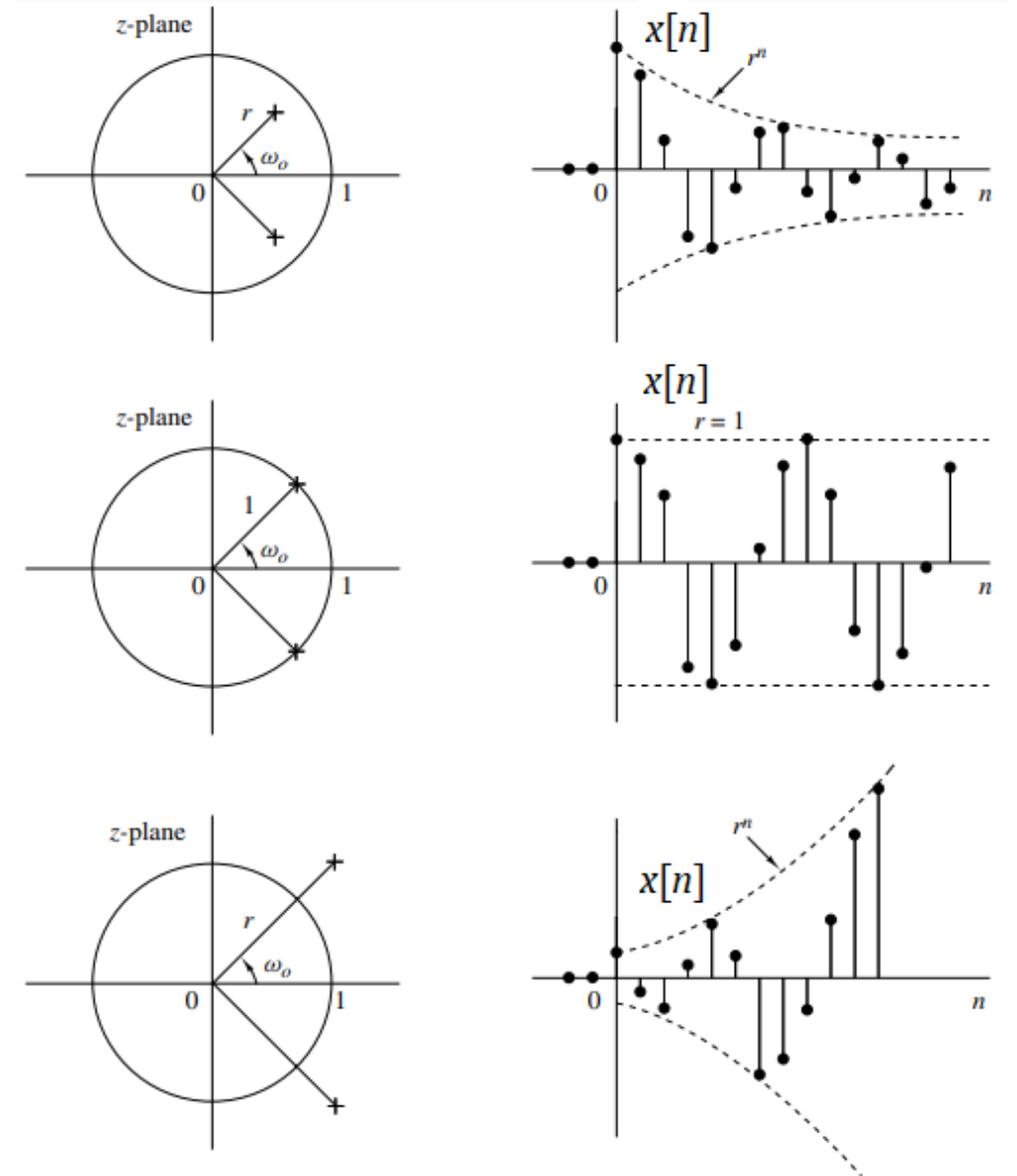
Some Common z-Transform Pairs

x[n]	X(z)	ROC
$\delta[n]$	1	Entire z-plane
$u[n]$	$\dfrac{1}{1 - z^{-1}}$	$|z| > 1$
$a^n u[n]$	$\dfrac{1}{1 - az^{-1}}$	$|z| > |a|$
$na^n u[n]$	$\dfrac{az^{-1}}{(1 - az^{-1})^2}$	$|z| > |a|$
$-a^n u[-n - 1]$	$\dfrac{1}{1 - az^{-1}}$	$|z| < |a|$
$-na^n u[-n - 1]$	$\dfrac{az^{-1}}{(1 - az^{-1})^2}$	$|z| < |a|$
$a^n \cos(\omega_0 n)\, u[n]$	$\dfrac{1 - az^{-1} \cos \omega_0}{1 - 2az^{-1} \cos \omega_0 + a^2 z^{-2}}$	$|z| > |a|$
$a^n \sin(\omega_0 n)\, u[n]$	$\dfrac{az^{-1} \sin \omega_0}{1 - 2az^{-1} \cos \omega_0 + a^2 z^{-2}}$	$|z| > |a|$

Inverse z-Transform

$$x[n] = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz$$

where C is a counterclockwise closed path encircling the origin and entirely in ROC.

Methods:

- Direct evaluation by contour integration

- Expansion into a series of terms, in the variables z

- Partial-fraction expansion and table lookup

Example: Find the inverse z-transform for

$$X(z) = \frac{1}{1 - 2z^{-1} + \frac{8}{9}z^{-2}} \; ; ROC: \; |z| > \frac{4}{3}$$

We get,

$$X(z) = \frac{z^2}{z^2 - 2z + \frac{8}{9}}$$

$$\Rightarrow X(z) = \frac{z^2}{\left(z - \frac{4}{3}\right)\left(z - \frac{2}{3}\right)}$$

$$\Rightarrow \frac{X(z)}{z} = \frac{z}{\left(z - \frac{4}{3}\right)\left(z - \frac{2}{3}\right)}$$

$$\Rightarrow \frac{X(z)}{z} = \frac{A}{z - \frac{4}{3}} + \frac{B}{z - \frac{2}{3}}$$

$$\therefore \frac{z}{\left(z - \frac{4}{3}\right)\left(z - \frac{2}{3}\right)} = \frac{A}{z - \frac{4}{3}} + \frac{B}{z - \frac{2}{3}}$$

$$\Rightarrow z = A\left(z - \frac{2}{3}\right) + B\left(z - \frac{4}{3}\right)$$

Putting $z = {}^2/_3$, we get $B = -1$

Putting $z = {}^4/_3$, we get $A = 2$

$$\therefore \frac{X(z)}{z} = \frac{2}{z - \frac{4}{3}} + \frac{-1}{z - \frac{2}{3}}$$

$$\Rightarrow X(z) = \frac{2}{1 - \frac{4}{3}z^{-1}} - \frac{1}{1 - \frac{2}{3}z^{-1}}$$

continued...

$$X(z) = \frac{2}{1-\frac{4}{3}z^{-1}} - \frac{1}{1-\frac{2}{3}z^{-1}}; ROC: |z| > \frac{4}{3} \qquad \text{--------------(i)}$$

So, x[n] is causal, and both terms in eqn (i) are causal terms.

$$\therefore x[n] = 2\left(\frac{4}{3}\right)^n u[n] - \left(\frac{2}{3}\right)^n u[n]$$

- What if $ROC: |z| < \frac{2}{3}$?

So, x[n] is anti-causal, and both terms in eqn (i) are anti-causal terms.

$$\therefore x[n] = -2\left(\frac{4}{3}\right)^n u[-n-1] + \left(\frac{2}{3}\right)^n u[-n-1]$$

continued...

$$X(z) = \frac{2}{1-\frac{4}{3}z^{-1}} - \frac{1}{1-\frac{2}{3}z^{-1}} \quad \text{--------------(i)}$$

- What if $ROC: \frac{2}{3} < |z| < \frac{4}{3}$?

So, x[n] is two-sided, and one term in eqn (i) is causal, another is anti-causal.

For ROC to exist (overlap):

- $|z| > \frac{2}{3}$ provides causal part.

- $|z| < \frac{4}{3}$ provides anti-causal part.

$$\therefore x[n] = -2\left(\frac{4}{3}\right)^n u[-n-1] - \left(\frac{2}{3}\right)^n u[n]$$

Example: Find the inverse z-transform for

$$X(z) = \frac{1}{(1 + z^{-1})(1 - z^{-1})^2}; x[n] \text{ is causal}$$

We get,

$$X(z) = \frac{z^3}{(z+1)(z-1)^2}$$

$$\Rightarrow \frac{X(z)}{z} = \frac{z^2}{(z+1)(z-1)^2}$$

$$\Rightarrow \frac{X(z)}{z} = \frac{A}{z+1} + \frac{B}{z-1} + \frac{C}{(z-1)^2}$$

$$\therefore \frac{z^2}{(z+1)(z-1)^2} = \frac{A}{z+1} + \frac{B}{z-1} + \frac{C}{(z-1)^2}$$

$$\Rightarrow z^2 = A(z-1)^2 + B(z+1)(z-1) + C(z+1)$$

After solving, $A = \frac{1}{4}; B = \frac{3}{4}; C = \frac{1}{2}$

$$\therefore \frac{X(z)}{z} = \frac{\frac{1}{4}}{z+1} + \frac{\frac{3}{4}}{z-1} + \frac{\frac{1}{2}}{(z-1)^2}$$

$$\Rightarrow X(z) = \frac{\frac{1}{4}}{1+z^{-1}} + \frac{\frac{3}{4}}{1-z^{-1}} + \frac{(\frac{1}{2})z^{-1}}{(1-z^{-1})^2}$$

$$\therefore x[n] = \frac{1}{4}(-1)^n u[n] + \frac{3}{4}(1)^n u[n] + \frac{1}{2}n(1)^n u[n]$$

$$\Rightarrow x[n] = \left(\frac{1}{4}(-1)^n + \frac{3}{4} + \frac{1}{2}n\right)u[n]$$

Example: Find the inverse z-transform for

$$X(z) = \frac{2z^{-1}}{1 - 2z^{-1} + 2z^{-2}}; x[n] \text{ is causal}$$

We get, $X(z) = \frac{2z}{z^2 - 2z + 2}$

$$\Rightarrow \frac{X(z)}{z} = \frac{2}{(z - (1 + j))(z - (1 - j))}$$

$$\Rightarrow \frac{X(z)}{z} = \frac{A}{z - (1 + j)} + \frac{A^*}{z - (1 - j)}$$

After solving, $A = -j; A^* = j$

$$\therefore \frac{X(z)}{z} = \frac{-j}{z - (1 + j)} + \frac{j}{z - (1 - j)}$$

$$\Rightarrow X(z) = \frac{-j}{1 - (1 + j)z^{-1}} + \frac{j}{1 - (1 - j)z^{-1}}$$

$$\therefore x[n] = (-j(1 + j)^n + j(1 - j)^n)u[n]$$

Formula: $Ap^n + A^*(p^*)^n = 2|A| \cdot |p|^n \cos(\omega n + \theta)$

where $A = |A|e^{j\theta}$ and $p = |p|e^{j\omega}$

$$\Rightarrow x[n] = \left(2(\sqrt{2})^n \cos\left(\frac{\pi}{4}n - \frac{\pi}{2}\right)\right)u[n]$$

$$\therefore x[n] = \left(2(\sqrt{2})^n \sin\left(\frac{\pi}{4}n\right)\right)u[n]$$

Note: Here, $A = -j = 1e^{-j\pi/2}$

$$p = 1 + j = \sqrt{2}e^{j\pi/4}$$

System Function $H(z)$

H(z) represents the z-domain characterization of a system, whereas h[n] is the corresponding time-domain characterization of the system.

$$y[n] = x[n] * h[n] \overset{z}{\leftrightarrow} Y(z) = X(z)H(z)$$

$$H(z) = \frac{Y(z)}{X(z)}$$

<u>Example:</u> Determine the system function and the impulse response of the causal system described by the difference equation: $y[n] = \frac{1}{5}y[n-1] + 7x[n]$

Computing the z-transform of this difference equation,

$$Y(z) = \frac{1}{5}z^{-1}Y(z) + 7X(z)$$

$$\Rightarrow Y(z)\left(1 - \frac{1}{5}z^{-1}\right) = 7X(z)$$

$$\Rightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{7}{1 - \frac{1}{5}z^{-1}}; ROC: |z| > \frac{1}{5}$$

$$\therefore h[n] = 7\left(\frac{1}{5}\right)^n u[n]$$

<u>Note:</u> This system has pole at $^1/_5$ and zero at origin.

Example: An LTI system has $h[n] = \delta[n] + 0.5\delta[n-1]$. Find the system's unit step response.

We get, $H(z) = 1 + 0.5z^{-1}; ROC: |z| > 0$

And, $x[n] = u[n] \overset{z}{\leftrightarrow} X(z) = \frac{1}{1-z^{-1}}; ROC: |z| > 1$

$$\therefore Y(z) = H(z)X(z) = \frac{1 + 0.5z^{-1}}{1 - z^{-1}}$$

$$\Rightarrow Y(z) = \frac{1}{1 - z^{-1}} + \frac{0.5z^{-1}}{1 - z^{-1}}; ROC: |z| > 1$$

$$\therefore y[n] = \textcolor{green}{u[n] + 0.5u[n-1]}$$

Example: Compute the step response of a causal LTI system with zero at 1, pole at 3, and H(0) = 4. Also find the LCCDE.

Let, $H(z) = k \cdot \frac{z-1}{z-3} \implies H(0) = 4 = k \cdot \frac{0-1}{0-3} \implies k = 12$

$$\therefore H(z) = 12 \cdot \frac{z-1}{z-3} = 12 \cdot \frac{1-z^{-1}}{1-3z^{-1}}; ROC: |z| > 3$$

And, $x[n] = u[n] \overset{z}{\leftrightarrow} X(z) = \frac{1}{1-z^{-1}}; ROC: |z| > 1$

$$\therefore Y(z) = H(z)X(z) = 12 \cdot \frac{1-z^{-1}}{1-3z^{-1}} \cdot \frac{1}{1-z^{-1}} = \frac{12}{1-3z^{-1}}; ROC: |z| > 3$$

$$\therefore y[n] = 12(3)^n u[n]$$

$$\frac{Y(z)}{X(z)} = H(z) = 12 \cdot \frac{1-z^{-1}}{1-3z^{-1}} = \frac{12-12z^{-1}}{1-3z^{-1}}$$

$$\therefore y[n] - 3y[n-1] = 12x[n] - 12x[n-1]$$

Causality & Stability with z-Transform

- An LTI system is causal if and only if the ROC of H(z) is the exterior of a circle of radius $r < \infty$, including the point $z = \infty$.

- An LTI system is BIBO stable if and only if the ROC of H(z) includes the unit circle.

A causal LTI system is BIBO stable if and only if all the poles of H(z) are inside the unit circle.

Example: An LTI system is characterized by the following system function:

$$H(z) = \frac{1}{1 - 3.5z^{-1} + 1.5z^{-2}}$$

System is non-causal. Specify ROC of H(z). Determine h[n]. Find stability.

After partial-fraction expansion, we get:

$$H(z) = \frac{-1/5}{1 - \frac{1}{2}z^{-1}} + \frac{6/5}{1 - 3z^{-1}}$$

Since the system is non-causal (given), $ROC: \frac{1}{2} < |z| < 3$

$$\therefore h[n] = -\frac{1}{5}\left(\frac{1}{2}\right)^n u[n] - \frac{6}{5}(3)^n u[-n-1]$$

ROC *contains* "unit circle", so the system is stable.

continued…

$$H(z) = \frac{-1/5}{1 - \frac{1}{2}z^{-1}} + \frac{6/5}{1 - 3z^{-1}}$$

What if the system is causal?

$$ROC: |z| > 3$$

$$\therefore h[n] = -\frac{1}{5}\left(\frac{1}{2}\right)^n u[n] + \frac{6}{5}(3)^n u[n]$$

ROC doesn't *contain* "unit circle", so the system is unstable.

What if the system is anti-causal?

$$ROC: |z| < \frac{1}{2}$$

$$\therefore h[n] = \frac{1}{5}\left(\frac{1}{2}\right)^n u[-n-1] - \frac{6}{5}(3)^n u[-n-1]$$

ROC doesn't *contain* "unit circle", so the system is unstable.

Pole-Zero Cancellation

When a z-transform has a pole that is at the same location as a zero, the pole is cancelled by the zero.

➢H(z) itself → order of the system reduced by one.

➢Product of H(z) and X(z) → pole of the system suppressed by zero in x[n], or vice versa.

By proper selection of the zeros of x[n], it is possible to suppress one or more system modes in the response of the system.

By proper selection of the zeros of H(z), it is possible to suppress one or more modes of x[n] from the response of the system.

Example: Find h[n] for a causal system is described by the following LCCDE:

$$y[n] = 2.5y[n-1] - y[n-2] + x[n] - 5x[n-1] + 6x[n-2]$$

We get, $y[n] - 2.5y[n-1] + y[n-2] = x[n] - 5x[n-1] + 6x[n-2]$

Applying z-transform,

$$Y(z) - 2.5z^{-1}Y(z) + z^{-2}Y(z) = X(z) - 5z^{-1}X(z) + 6z^{-2}X(z)$$

$$\Rightarrow Y(z)(1 - 2.5z^{-1} + z^{-2}) = X(z)(1 - 5z^{-1} + 6z^{-2})$$

$$\Rightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{1 - 5z^{-1} + 6z^{-2}}{1 - 2.5z^{-1} + z^{-2}}$$

$$\Rightarrow H(z) = \frac{z^2 - 5z + 6}{z^2 - 2.5z + 1} = \frac{(z-2)(z-3)}{(z-2)(z-0.5)}$$

$$\Rightarrow H(z) = \frac{z-3}{z-0.5} = \frac{z - 0.5 - 2.5}{z - 0.5} = 1 - \frac{2.5}{z - 0.5}$$

$$\Rightarrow H(z) = 1 - \frac{2.5z^{-1}}{1 - 0.5z^{-1}}; ROC: |z| > \frac{1}{2}$$

$$\therefore h[n] = \delta[n] - 2.5(0.5)^{n-1}u[n-1]$$

Example: A causal LTI system is described by:

$$y[n] = \frac{5}{6}y[n-1] - \frac{1}{6}y[n-2] + x[n]$$

Find the input signal that will cancel the nearest pole to the origin. Also find system response for that input.

$$Y(z)\left(1 - \frac{5}{6}z^{-1} + \frac{1}{6}z^{-2}\right) = X(z)$$

$$\therefore H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - \frac{5}{6}z^{-1} + \frac{1}{6}z^{-2}}$$

$$\Rightarrow H(z) = \frac{1}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - \frac{1}{3}z^{-1}\right)}$$

This system has two real poles at $^1/_2$ and $^1/_3$

System $ROC: |z| > \frac{1}{2}$

The nearest pole is at $^1/_3$ (to be cancelled).

So, X(z) must be $\left(1 - \frac{1}{3}z^{-1}\right)$; ROC: $|z| > 0$

$$\therefore x[n] = \delta[n] - \frac{1}{3}\delta[n-1]$$

Now, $Y(z) = H(z)X(z) = \frac{1}{1-\frac{1}{2}z^{-1}}$; $ROC: |z| > \frac{1}{2}$

$$\therefore y[n] = \left(\frac{1}{2}\right)^n u[n]$$

continued…

- What if you want to cancel the farthest pole from the origin?

$$H(z) = \frac{1}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - \frac{1}{3}z^{-1}\right)}$$

The farthest pole is at $^1/_2$ (to be cancelled).

So, X(z) must be $\left(1 - \frac{1}{2}z^{-1}\right); ROC: |z| > 0$

$$\therefore x[n] = \delta[n] - \frac{1}{2}\delta[n-1]$$

Now, $Y(z) = H(z)X(z) = \frac{1}{1 - \frac{1}{3}z^{-1}}; ROC: |z| > \frac{1}{3}$

$$\therefore y[n] = \left(\frac{1}{3}\right)^n u[n]$$

Example: A cell phone signal $x[n]$ is distorted by multipath reflections off of city buildings. What your cell phone receives is not $x[n]$ but $y[n]$, where $y[n] = x[n] - 0.75x[n-1] + 0.125x[n-2]$. Can you find a filter that recovers $x[n]$ from $y[n]$?

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n] \longrightarrow \boxed{g[n]} \longrightarrow x[n]$$

We need to compute the inverse filter $g[n]$ that can undo the effect of $h[n]$.

$$H(z) = \frac{Y(z)}{X(z)} = 1 - 0.75z^{-1} + 0.125z^{-2}$$

$$\Longrightarrow H(z) = \frac{z^2 - 0.75z + 0.125}{z^2}$$

We want $h[n] * g[n] = \delta[n]$, which implies $H(z)G(z) = 1$.

$$G(z) = \frac{1}{H(z)} = \frac{z^2}{z^2 - 0.75z + 0.125}$$

continued...

$$\Rightarrow G(z) = \frac{z^2}{(z - 0.5)(z - 0.25)}$$

$$\Rightarrow \frac{G(z)}{z} = \frac{z}{(z - 0.5)(z - 0.25)}$$

$$\Rightarrow \frac{G(z)}{z} = \frac{2}{z - 0.5} + \frac{-1}{z - 0.25}$$

$$\Rightarrow G(z) = \frac{2z}{z - 0.5} - \frac{z}{z - 0.25}$$

$$\Rightarrow G(z) = \frac{2}{1 - 0.5z^{-1}} - \frac{1}{1 - 0.25z^{-1}}$$

$$\therefore g[n] = 2(0.5)^n u[n] - (0.25)^n u[n]$$

Digital Filter

Use:

- Signal separation (e.g. EKG contaminated with noise)
- Signal restoration (e.g. image captured with shaky camera)

Analog vs Digital

- Analog filters are cheap, fast, and have a large dynamic range.
- Digital filters are vastly superior in the level of performance.

Digital Filter Type

i. FIR Filter: has Finite Impulse Response [carried out by convolution]

ii. IIR Filter: has Infinite Impulse Response [carried out by recursion]

Filter Parameters

Every linear filter has

- impulse response

- step response

- frequency response

Some info about dB:

Every 20 dB means that the amplitude has changed by a factor of 10.

-3 dB means that the amplitude is reduced to 0.707, and the power is reduced to 0.5.
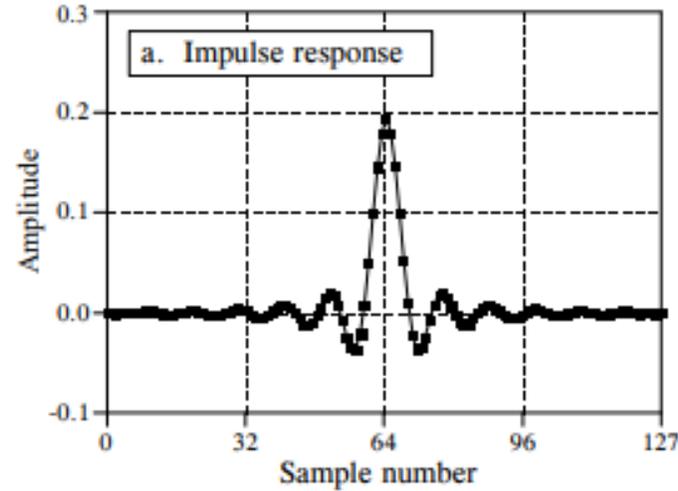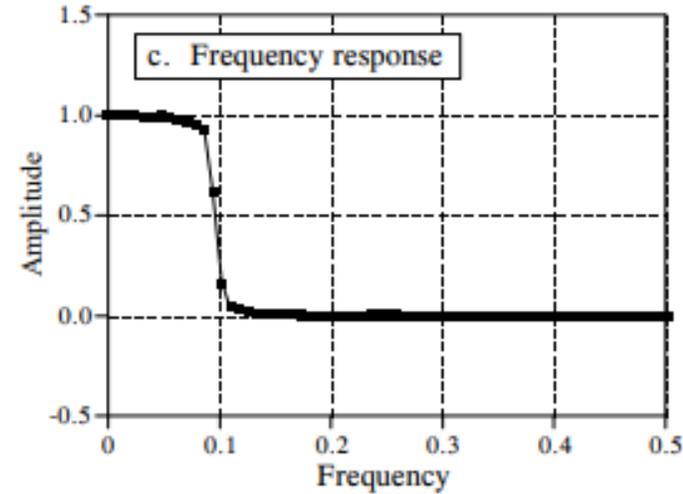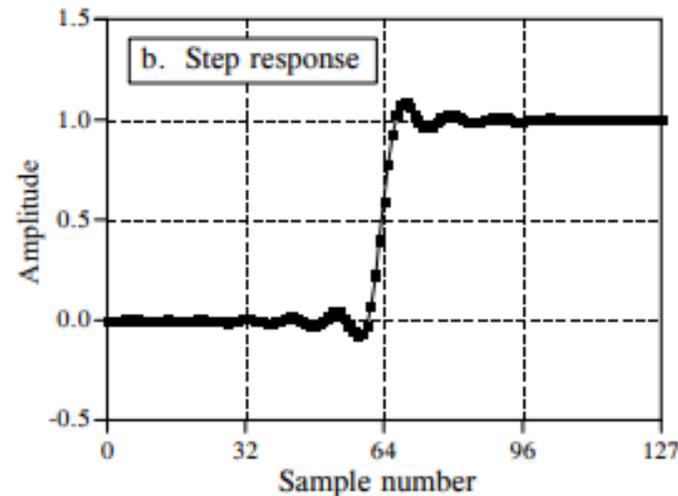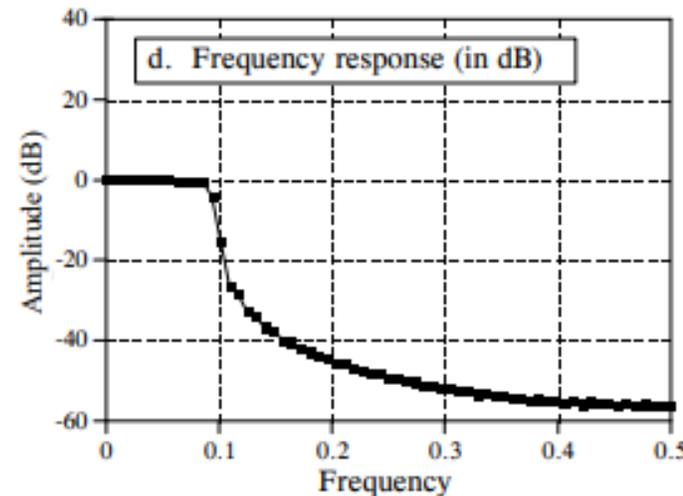
Information is Represented in Signals

- Step response describes how information represented in the time domain is being modified by the system.

- Frequency response shows how information represented in the frequency domain is being changed.

Good performance in the time domain results in poor performance in the frequency domain, and vice versa.

Example:

Filter for noise removal from an EKG: step response is the important parameter, and frequency response is of little concern.

Filter for a hearing aid: frequency response is all important, while step response doesn't matter.

Time Domain Parameters of Digital Filter
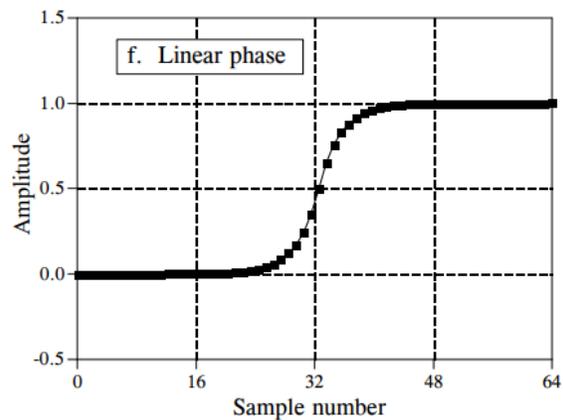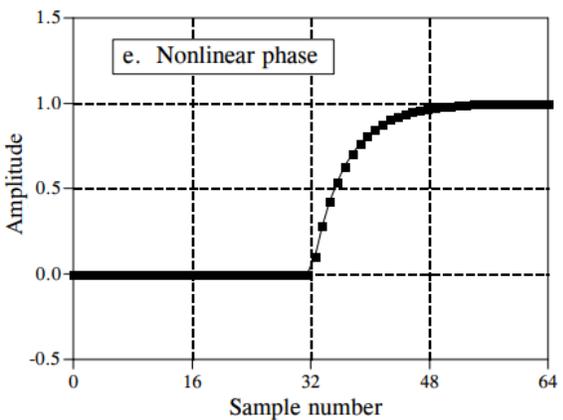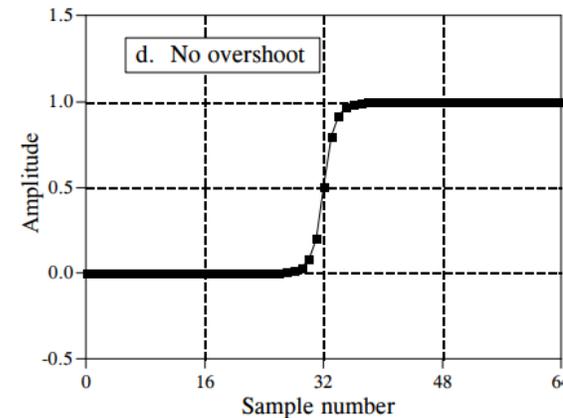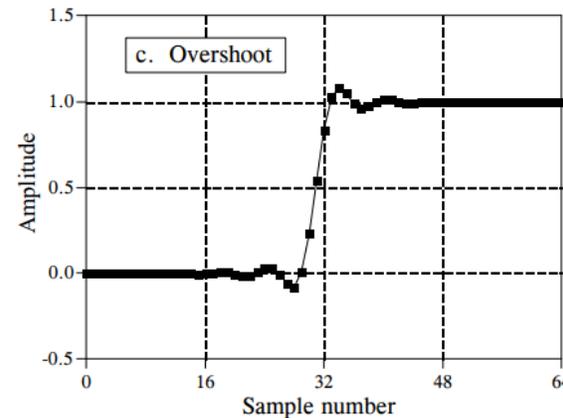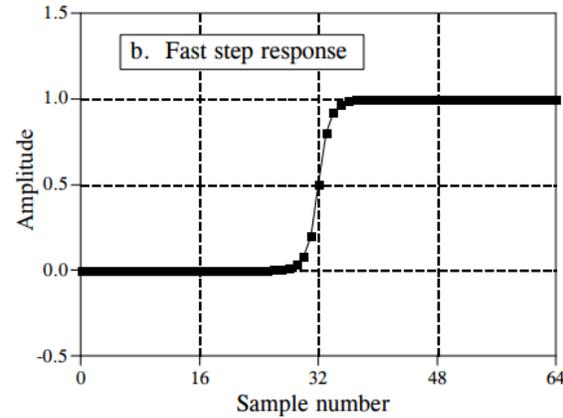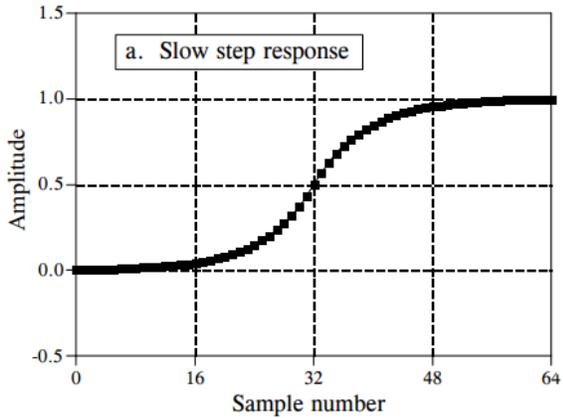
Better:
- ✓ Fast step response
- ✓ No overshoot
- ✓ Linear Phase

Frequency Domain Parameters of Digital Filter
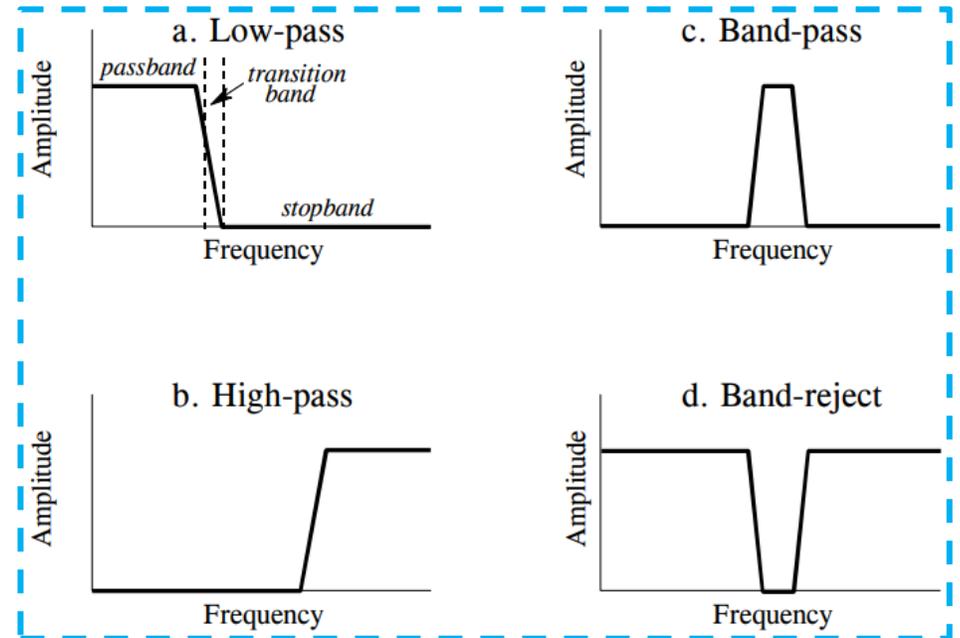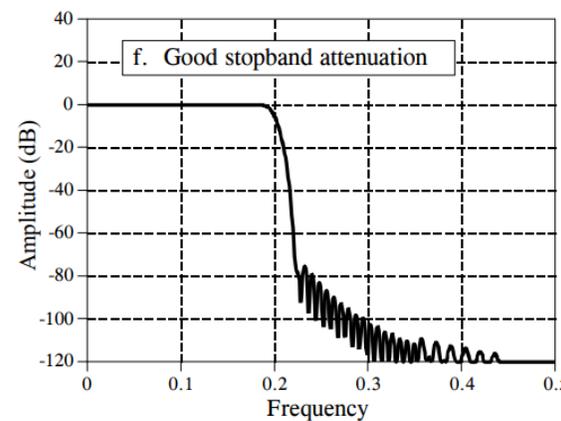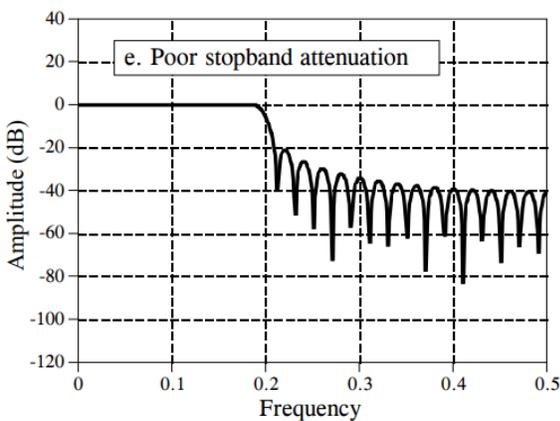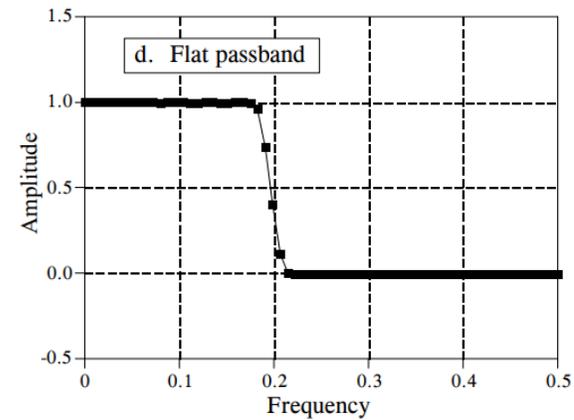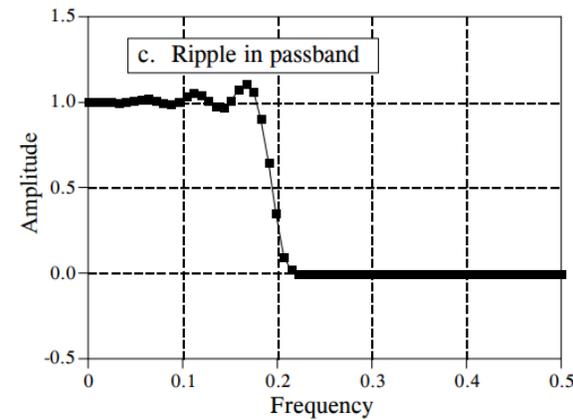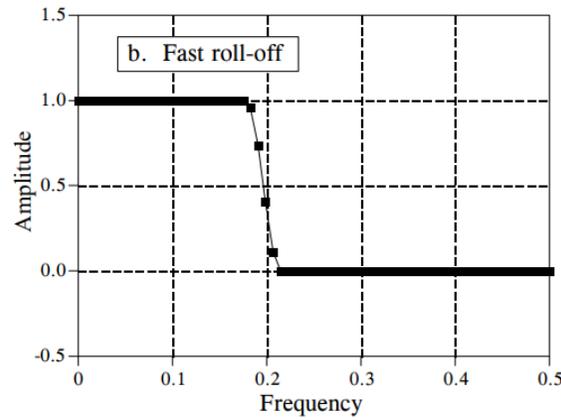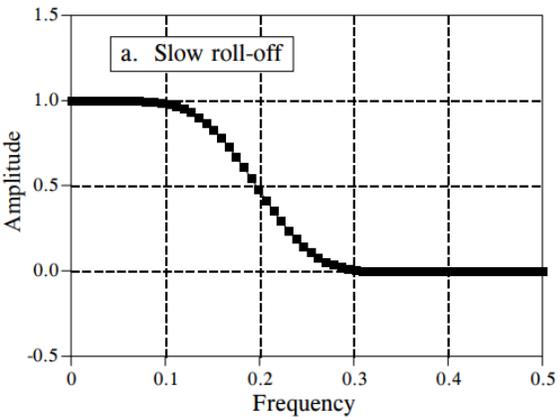
Better:
- ✓ Fast roll-off
- ✓ Flat passband
- ✓ Good stopband attenuation

Low-Pass Filter to High-Pass Filter

a. High-pass by adding parallel stages

Low-pass

All-pass
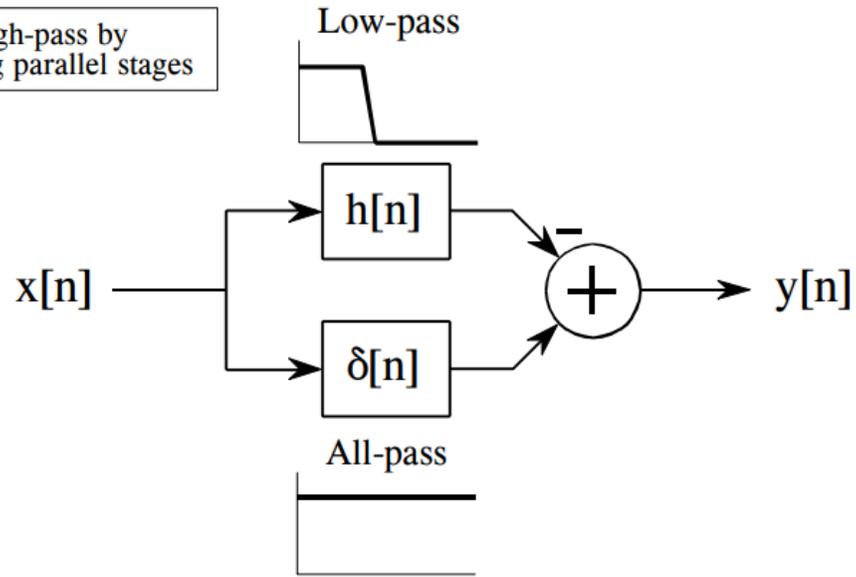
b. High-pass in a single stage

High-pass
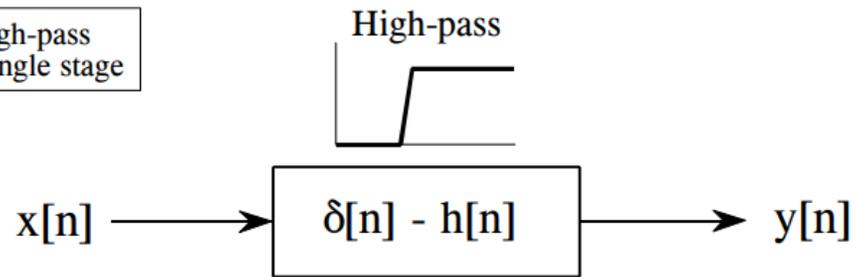
Time Domain

Frequency Domain

a. Original filter kernel

b. Original frequency response

c. Filter kernel with spectral inversion

d. Inverted frequency response

Flipped top-for-bottom
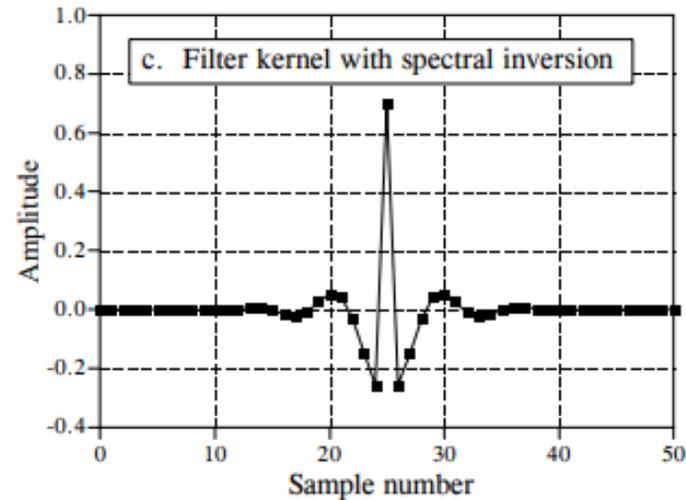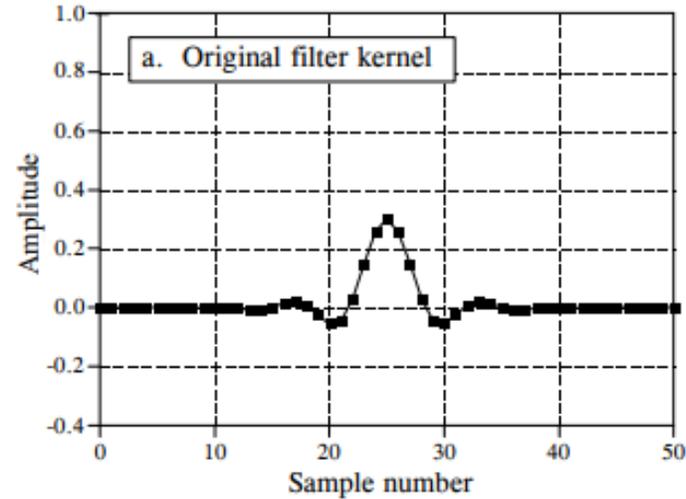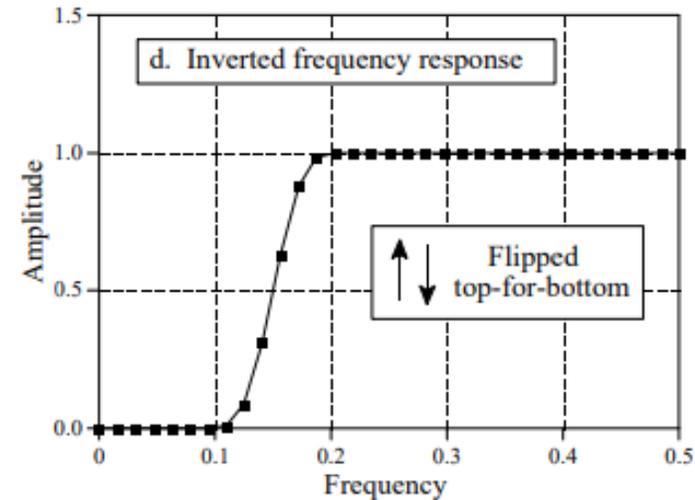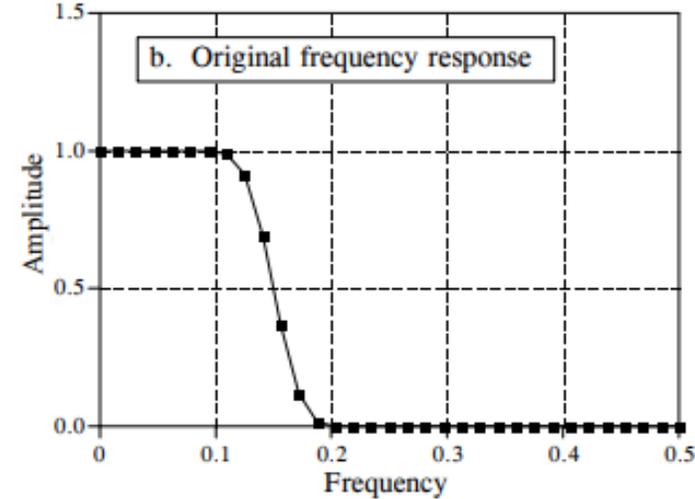
Band-Pass & Band-Reject Filter from LPF & HPF
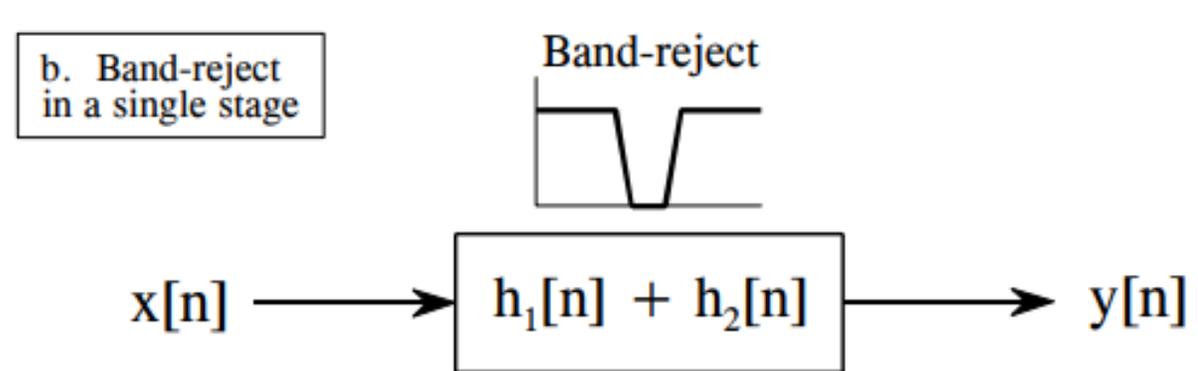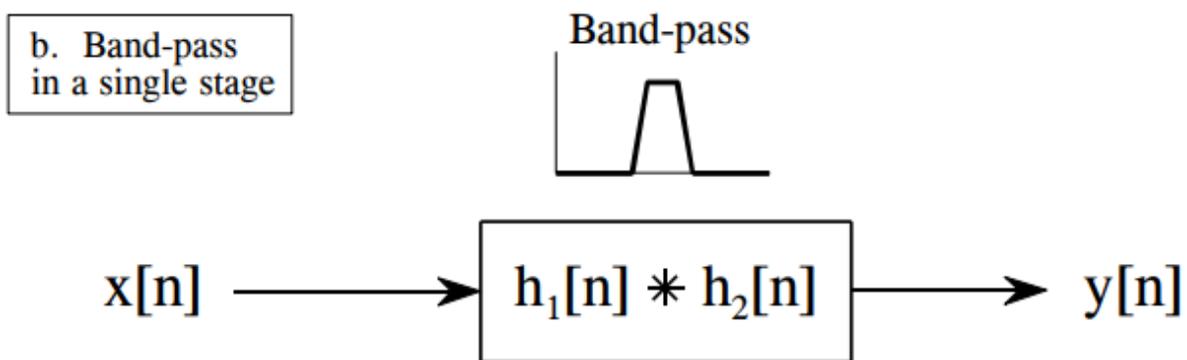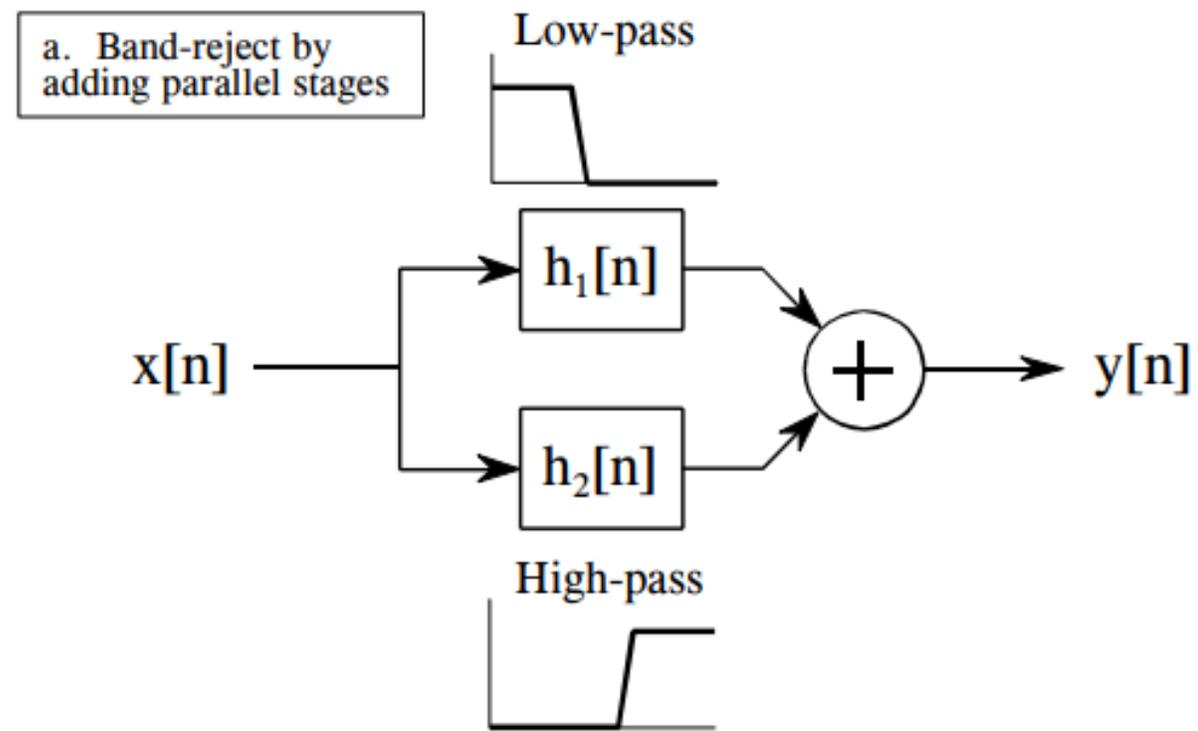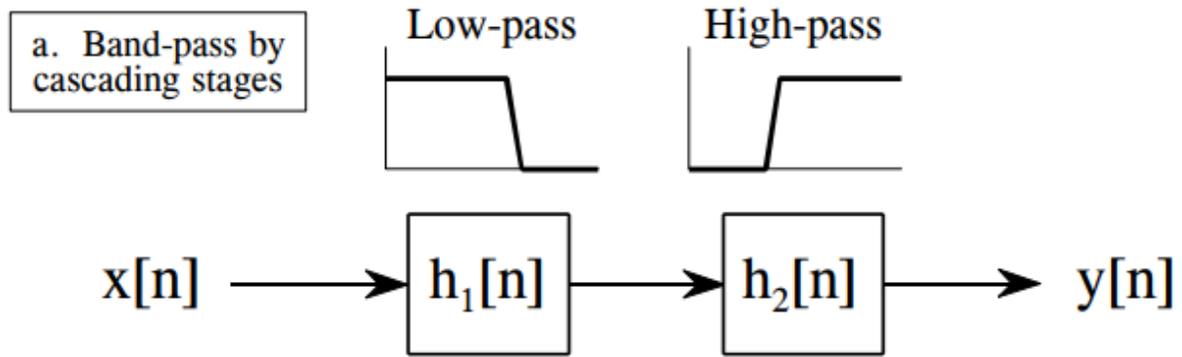
Moving Average Filter

$$y[i] = \frac{1}{M} \sum_{j=0}^{M-1} x[i+j]$$

For example, a 5-point MA filter:

$$y[80] = \frac{x[80] + x[81] + x[82] + x[83] + x[84]}{5}$$

Alternatively,
$$y[80] = \frac{x[78] + x[79] + x[80] + x[81] + x[82]}{5}$$

A 5-point filter has the filter kernel: $[\ldots 0\; 0\; \frac{1}{5}\; \frac{1}{5}\; \frac{1}{5}\; \frac{1}{5}\; \frac{1}{5}\; 0\; 0 \ldots]$

That is, MA filter is a convolution of the *input signal* with a rectangular pulse having an area of unity.

Remember: MA filter is an FIR filter.

a. Original signal

b. 11 point moving average

c. 51 point moving average
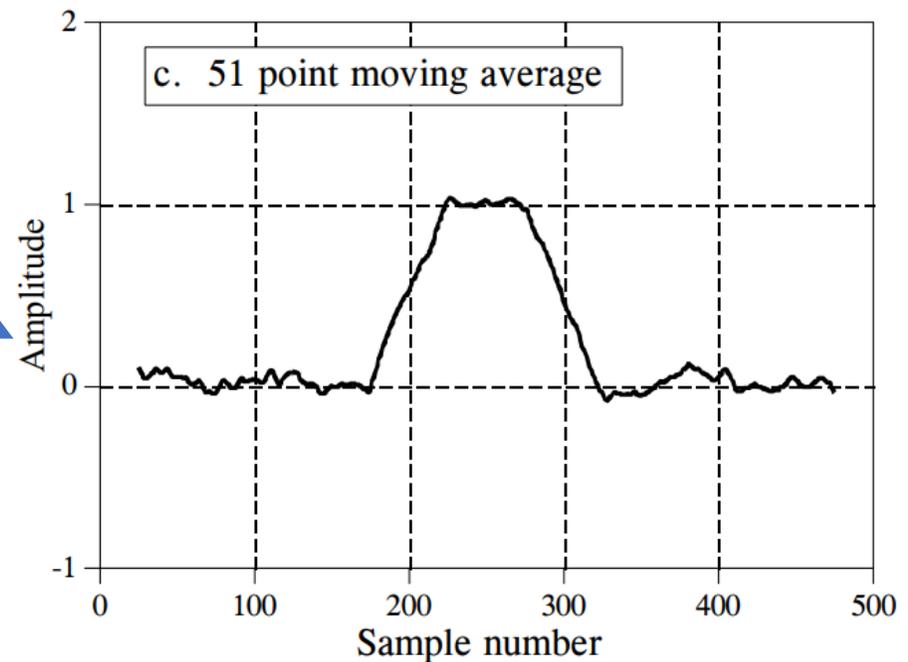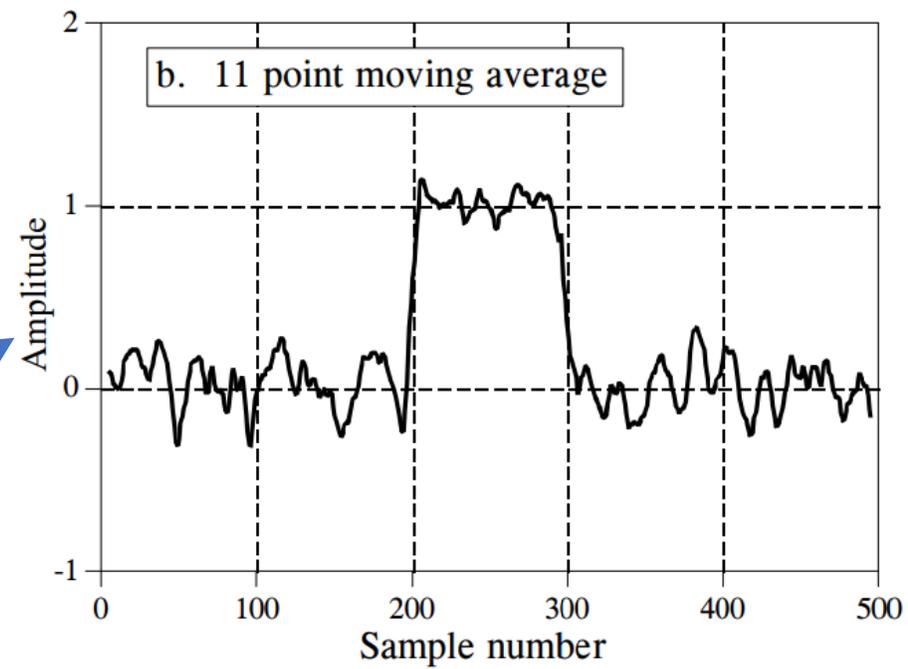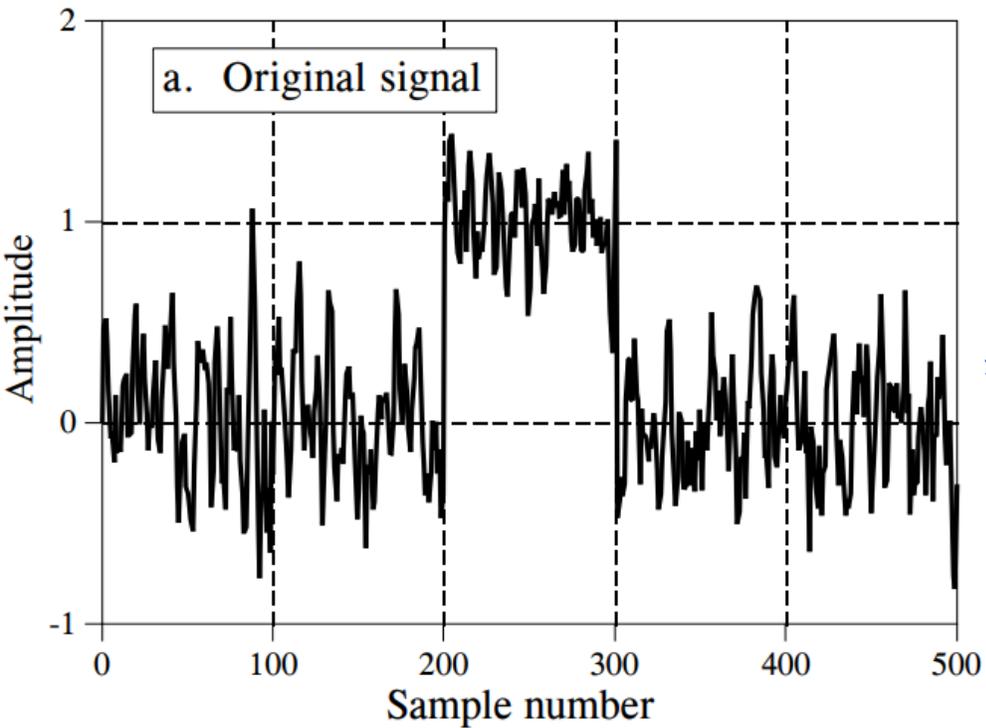
The amount of noise reduction is equal to the square-root of the number of points in the average.

For example, a 100 point moving average filter reduces the noise by a factor of 10.

Frequency Response of MA Filter

- MA is an exceptionally good smoothing filter.
- But, MA is an exceptionally bad low-pass filter.

Frequency response of an M-point MA filter:

$$H[f] = \frac{1}{M} \frac{\sin(\pi f M)}{\sin(\pi f)}$$
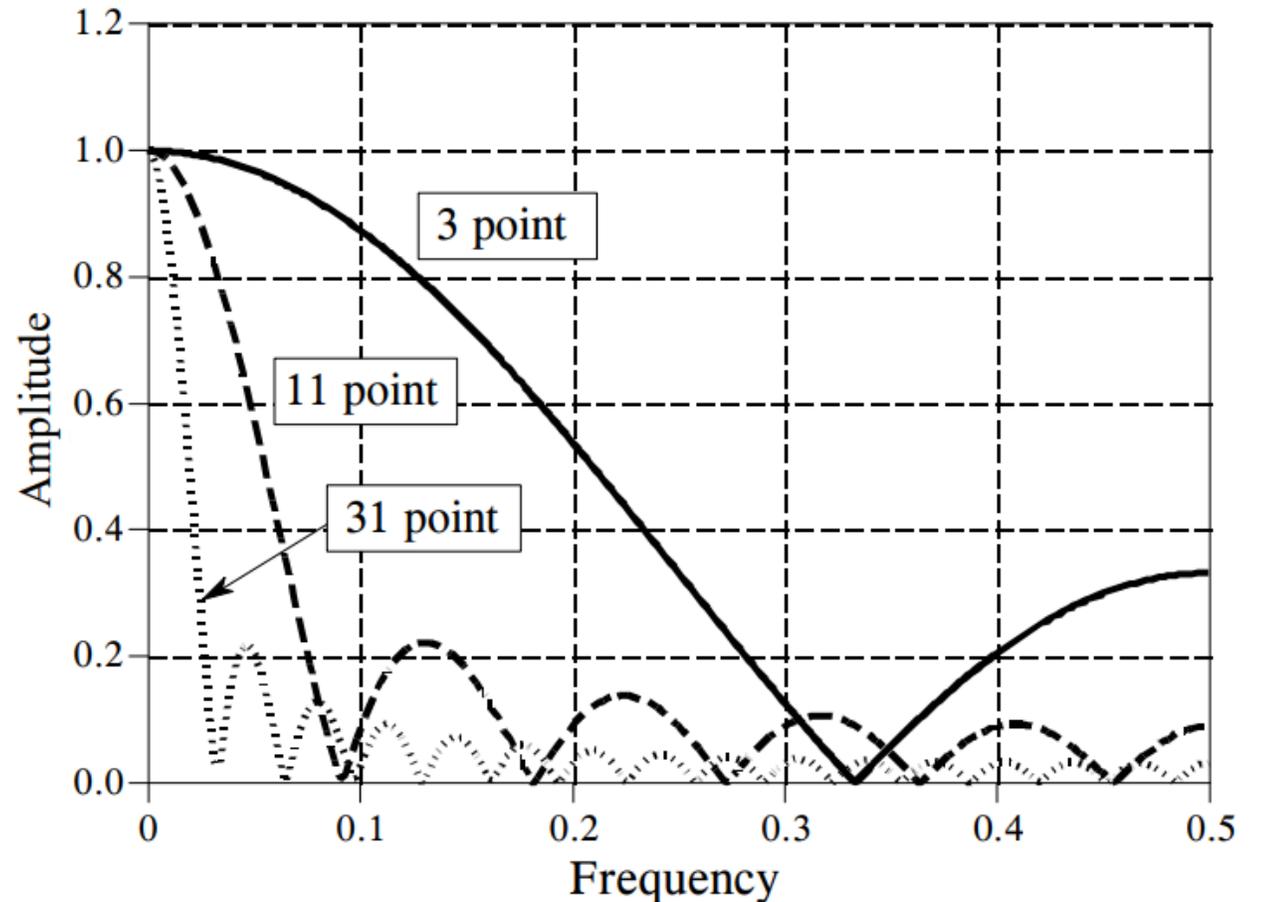
Windowed-Sinc Filter

Time Domain

b. Ideal filter kernel

c. Truncated-sinc filter kernel

abrupt end

Frequency Domain

a. Ideal frequency response
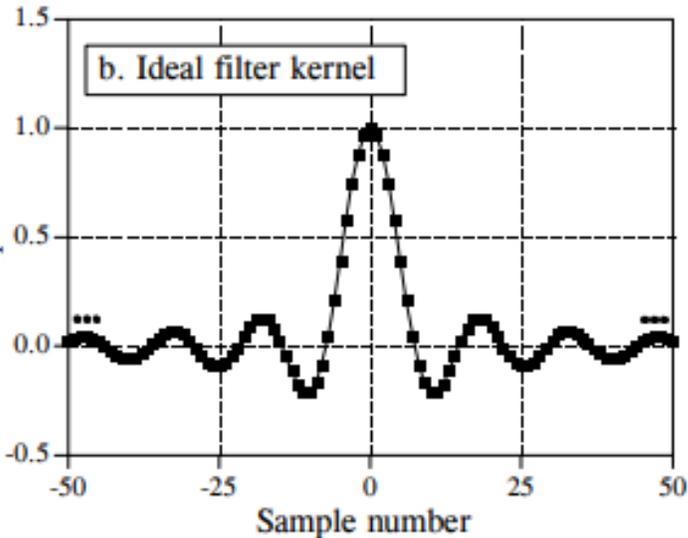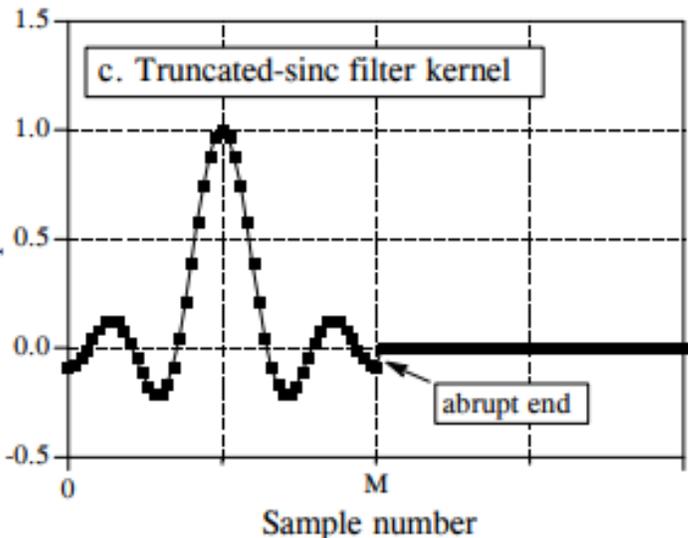
d. Truncated-sinc frequency response

While this infinite length (in time domain) is not a problem for mathematics, it is a show stopper for computers.
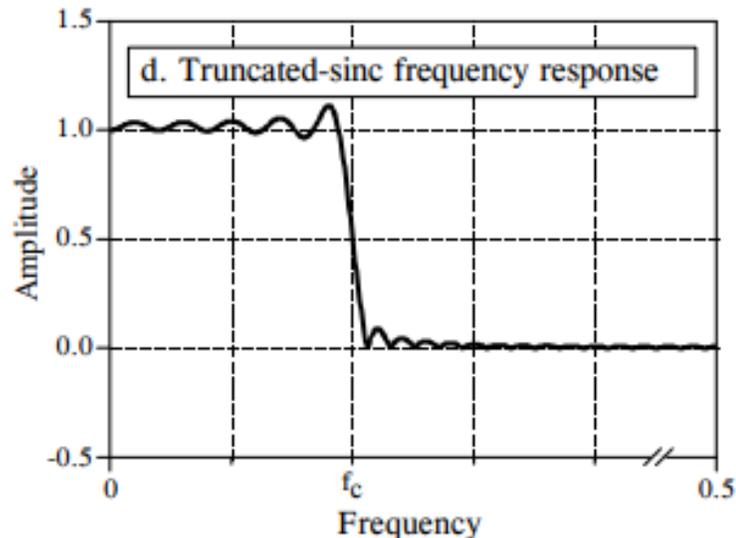
Now there is excessive ripple in the passband and poor attenuation in the stopband.
Increasing the length of the filter kernel does not reduce these problems.

Windowing comes to the rescue!

e. Blackman or Hamming window
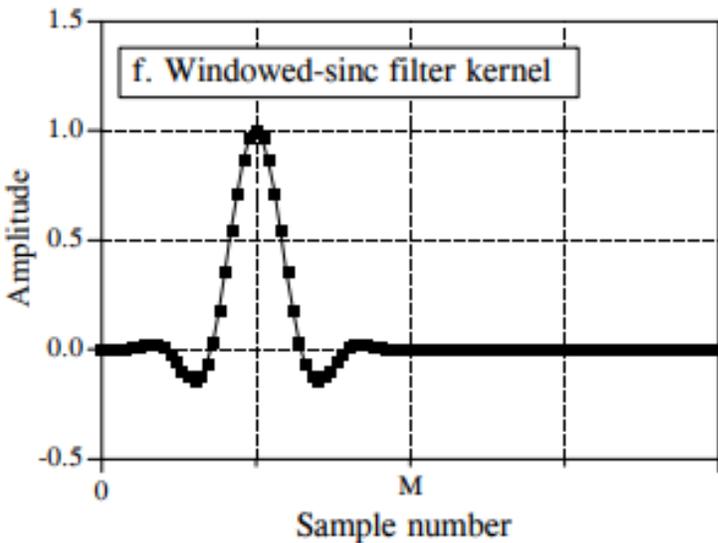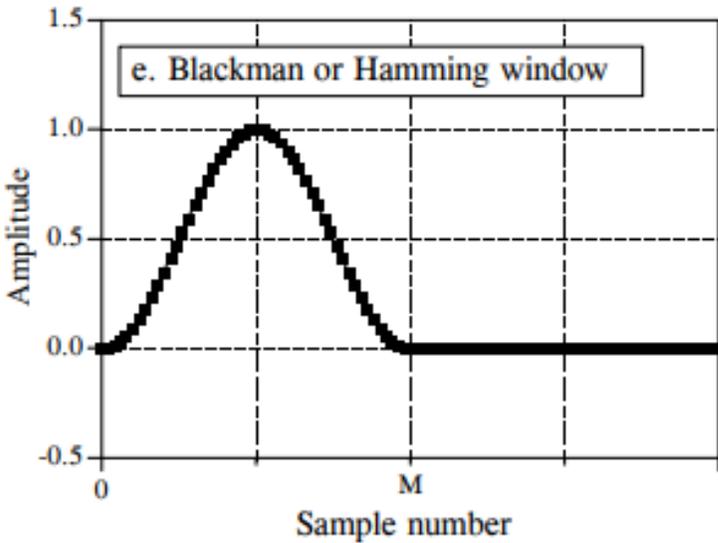
Multiplying the truncated-sinc (c), by the Blackman or Hamming window (e), results in the windowed-sinc filter kernel shown in (f).

f. Windowed-sinc filter kernel

g. Windowed-sinc frequency response
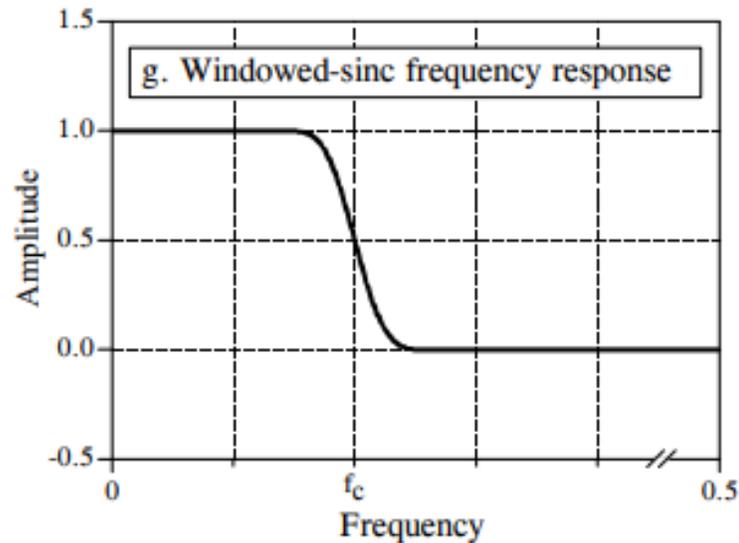
The passband is now flat, and the stopband attenuation is so good it cannot be seen in this graph.

Window: Hamming vs Blackman

- Hamming Window:

$$w[n] = 0.54 - 0.46 \cos\left(\frac{2\pi n}{M}\right)$$

- Blackman Window:

$$w[n] = 0.42 - 0.5 \cos\left(\frac{2\pi n}{M}\right) + 0.08 \cos\left(\frac{4\pi n}{M}\right)$$

Comparison:

Hamming has about a 20% faster roll-off than Blackman.

Blackman has a better stopband attenuation (-74 dB) than Hamming (-53 dB).

Blackman has a passband ripple of about 0.02%, while Hamming is typically 0.2%.

Conclusion:

In general, Blackman should be your first choice; a slow roll-off is easier to handle than poor stopband attenuation.
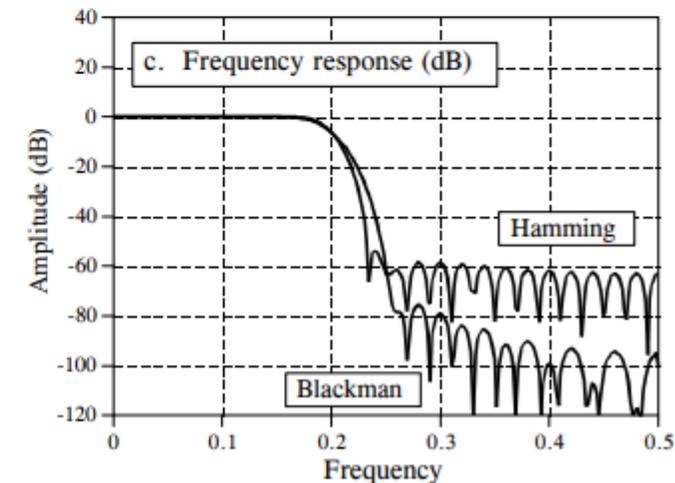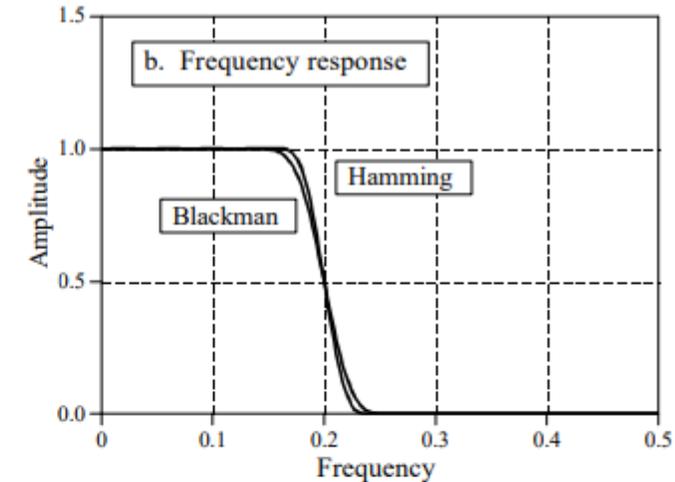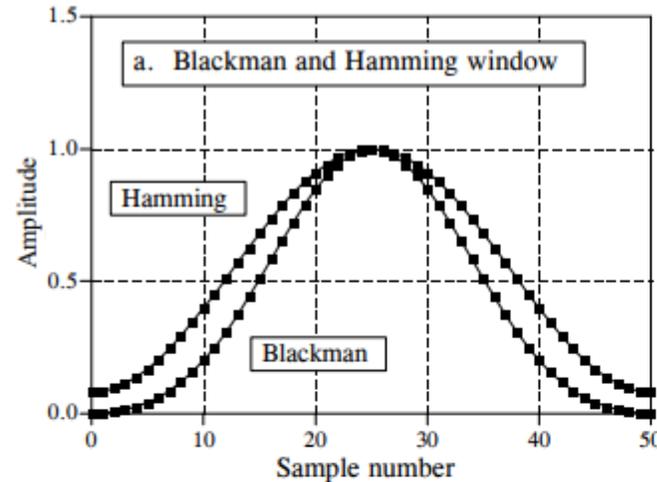
Designing the Windowed-Sinc Filter

Parameters:

➤ Cutoff frequency, f_C

➤ Length of the filter kernel, $M + 1$

Value for M sets the roll-off:

$$BW_{\text{tran}} \approx \frac{4}{M}$$

Both f_C and BW_{tran} are expressed as fraction of the sampling rate. Thus, they must be between 0 and 0.5.

f_C is measured at the one-half amplitude point.

a. Roll-off vs. kernel length

M=40
M=200
M=20

b. Roll-off vs. cutoff frequency

M=60
f_C=0.05
f_C=0.25
f_C=0.45
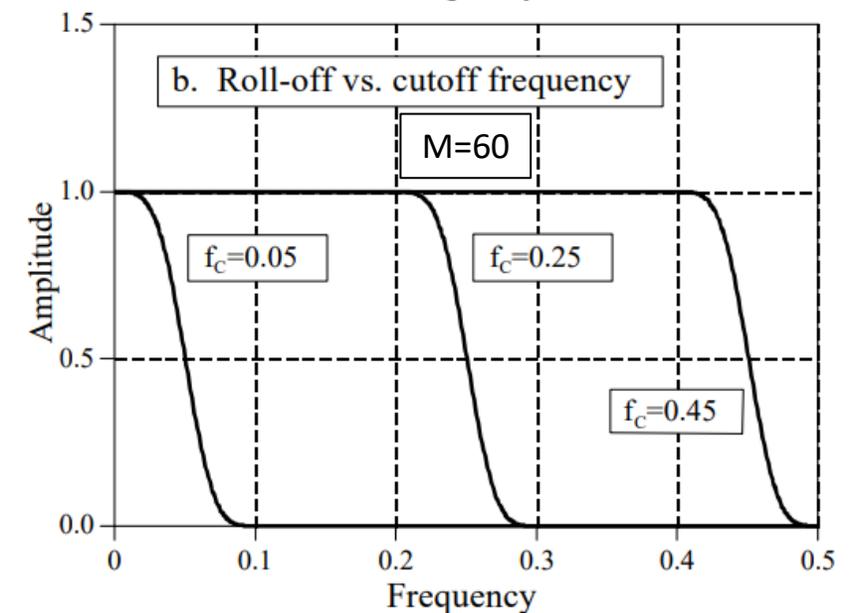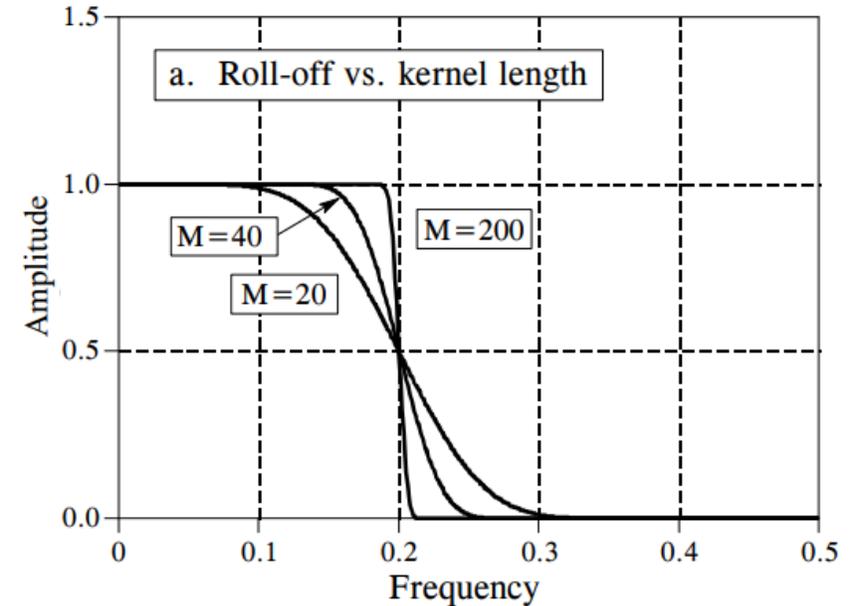
continued...

Filter Kernel: (*Blackman LPF*)

$$h[n] = K \operatorname{sinc}\left(2\pi f_C\left(n - \frac{M}{2}\right)\right) \cdot \left(0.42 - 0.5\cos\left(\frac{2\pi n}{M}\right) + 0.08\cos\left(\frac{4\pi n}{M}\right)\right)$$

$$\text{for } 0 \leq n \leq M$$

K is chosen to provide unity gain at zero frequency (normalizing coefficient).

M must be an even integer.

Filter kernel — a. $f_C = 0.015$, $M = 500$

Step response — b. $f_C = 0.015$, $M = 500$

Filter kernel — c. $f_C = 0.04$, $M = 500$

Step response — d. $f_C = 0.04$, $M = 500$
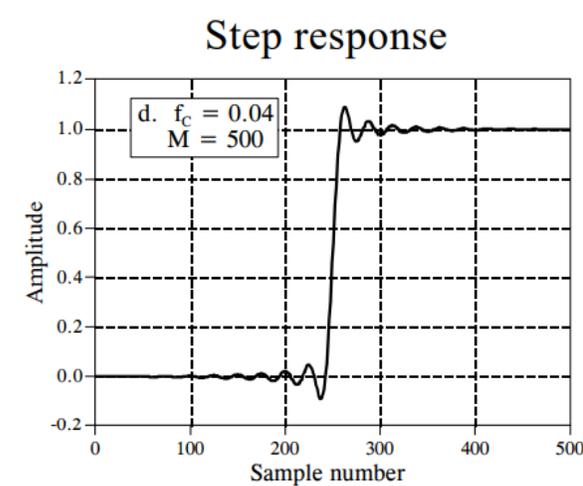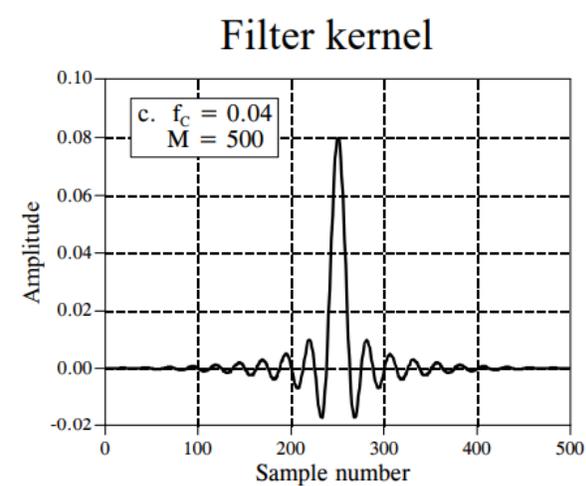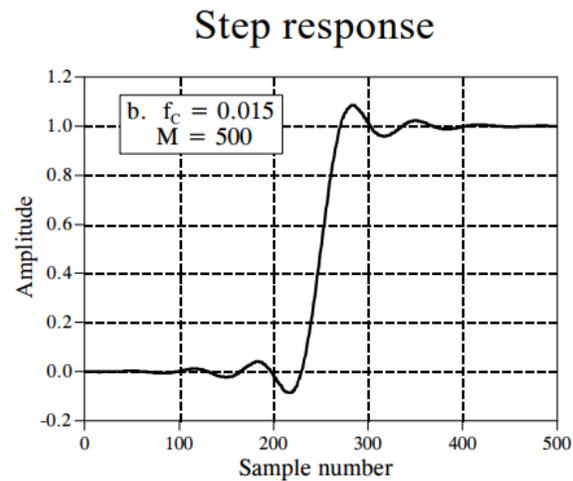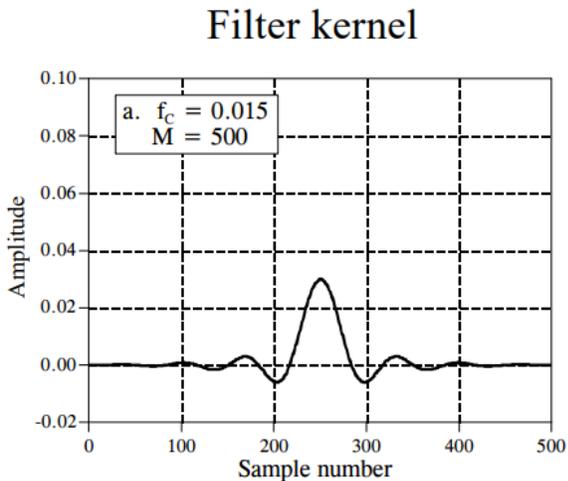
Example: EEG pattern containing alpha rhythm occurs between 7 and 12 Hz, and beta rhythm occurs between 17 and 20 Hz. Design a Blackman LPF that can separate alpha from beta rhythms. The EEG signal was digitized at a sampling rate of 100 sample/second. Set your transition bandwidth at 4 Hz.

Let, f_C = 14 Hz = 0.14 of sampling rate.

Given, BW_{tran} = 4 Hz = 0.04 of sampling rate.

$$\therefore M = \frac{4}{BW_{tran}} = \frac{4}{0.04} = 100$$

a. Low-pass filter
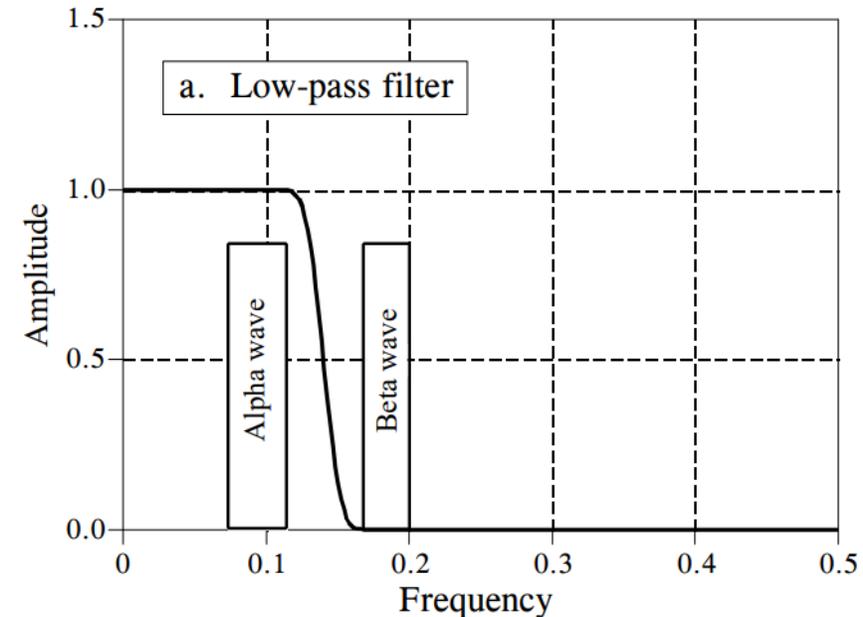
$$h[n] = K \operatorname{sinc}\left(2\pi f_C\left(n - \frac{M}{2}\right)\right) \cdot \left(0.42 - 0.5\cos\left(\frac{2\pi n}{M}\right) + 0.08\cos\left(\frac{4\pi n}{M}\right)\right)$$

$$= K \operatorname{sinc}\left(0.28\pi(n - 50)\right) \cdot \left(0.42 - 0.5\cos\left(\frac{2\pi n}{100}\right) + 0.08\cos\left(\frac{4\pi n}{100}\right)\right)$$

for $0 \leq n \leq 100$

Kaiser Window

$$\delta = min\{\delta_p, \delta_s\}$$

$$A = -20 \log_{10} \delta$$

$$\beta = \begin{cases} 0.1102(A - 8.7), & A > 50 \\ 0.5842(A - 21)^{0.4} + 0.07886(A - 21), & 21 \leq A \leq 50 \\ 0, & A < 21 \end{cases}$$

$$M = \text{even} \left[\frac{A - 8}{2.285(\omega_s - \omega_p)} \right]$$

- $M + 1$ is the length of the window.
- β is the shape parameter.

Large values of β result in reduced ripple.
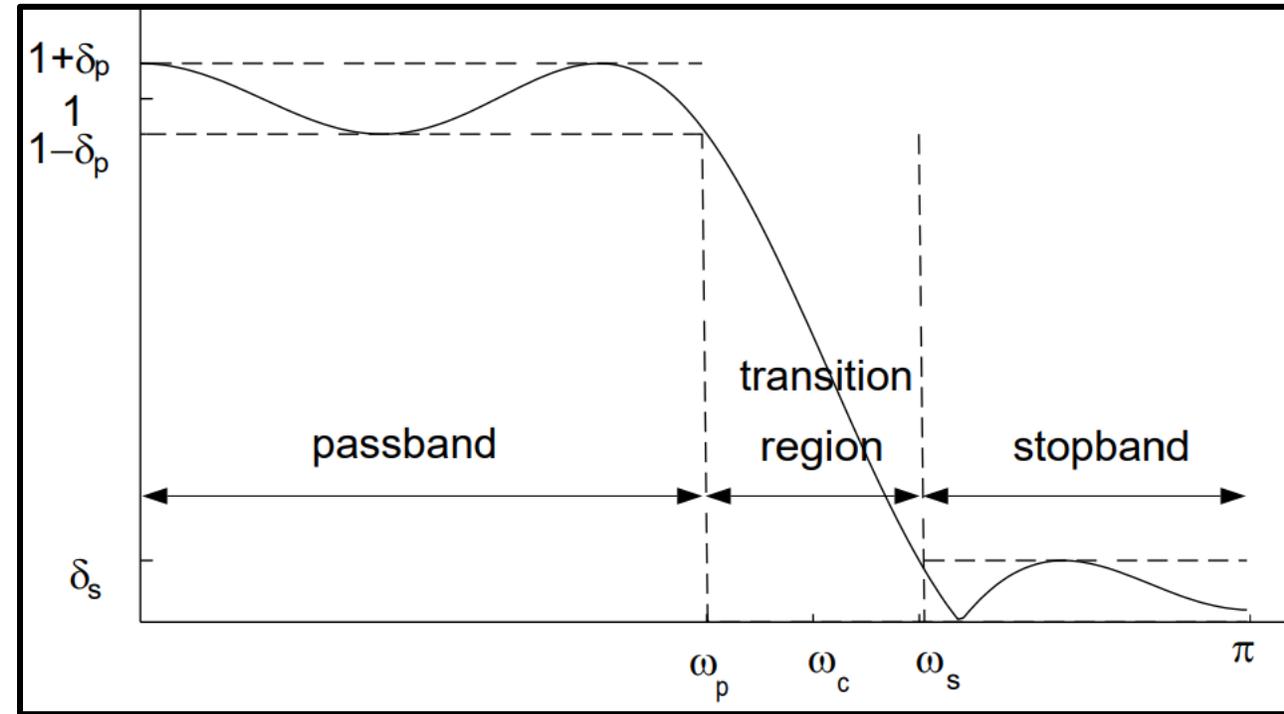
continued...

$$w[n] = \begin{cases} \dfrac{I_0\left(\beta \sqrt{1 - \left(\dfrac{n - 0.5M}{0.5M}\right)^2} \right)}{I_0(\beta)}, & 0 \le n \le M \\ 0, & otherwise \end{cases}$$
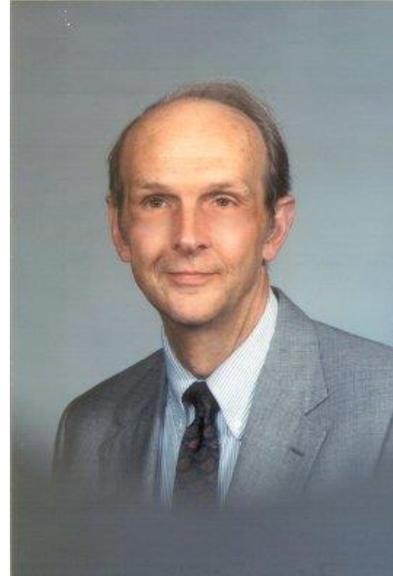
James Frederick Kaiser

where $I_0(\cdot)$ is the 0th-order modified Bessel function of the 1st kind, that can be easily generated using:

$$I_0(x) = 1 + \sum_{k=1}^{k=\infty} \left(\frac{(0.5x)^k}{k!} \right)^2$$

✓ A Kaiser Window is nearly optimum in the sense of having the most energy in its main-lobe for a given side-lobe amplitude.

Example: Design an FIR LPF using Kaiser window according to the following specifications

$$0.99 \leq \left| H\left(e^{j\omega}\right) \right| \leq 1.01 \quad 0 \leq |\omega| \leq 0.19\pi$$

$$\left| H\left(e^{j\omega}\right) \right| \leq 0.01 \qquad 0.21\pi \leq |\omega| \leq \pi$$

Solve:

$$\delta = min\{\delta_p, \delta_s\} = min\{0.01, 0.01\} = 0.01$$

$$A = -20 \log_{10} \delta = -20 \log_{10} 0.01 = 40$$

$$\beta = 0.5842(A - 21)^{0.4} + 0.07886(A - 21) = 0.5842(19)^{0.4} + 0.07886(19)$$

$$\implies \beta = 3.395$$

$$M = \text{even} \left[\frac{A - 8}{2.285(\omega_s - \omega_p)} \right] = \text{even} \left[\frac{40 - 8}{2.285(0.21\pi - 0.19\pi)} \right]$$

$$= \text{even}[222.8] = 224$$

continued...

$$\therefore w[n] = \begin{cases} \dfrac{I_0\left(\beta\sqrt{1 - \left(\dfrac{n - 0.5M}{0.5M}\right)^2}\right)}{I_0(\beta)}, & 0 \leq n \leq M \\ 0, & otherwise \end{cases}$$

$$\Rightarrow w[n] = \begin{cases} \dfrac{I_0\left(0.0304\sqrt{224n - n^2}\right)}{I_0(3.395)}, & 0 \leq n \leq 224 \\ 0, & otherwise \end{cases}$$

$$\therefore h[n] = h_d[n] \cdot w[n]$$

$$\text{where } h_d[n] = K\,\text{sinc}\big(0.2\pi(n - 112)\big)$$

$$\omega_C = \frac{\omega_p + \omega_s}{2} = \frac{0.19\pi + 0.21\pi}{2}$$

$$\Rightarrow \omega_C = 0.2\pi$$

IIR (Recursive) Filter

$$y[n] = b_0 x[n] + b_1 x[n-1] + b_2 x[n-2] + b_3 x[n-3] + \cdots$$
$$+ a_1 y[n-1] + a_2 y[n-2] + a_3 y[n-3] + \cdots$$

$a_1, a_2, a_3, \ldots, b_0, b_1, b_2, b_3, \ldots$ are called Recursion Coefficients.

IIR filters execute very rapidly, but have less performance and flexibility than other digital filters.

In theory, recursive filter convolves the input signal with a very **long** filter kernel; although only a few coefficients are involved.

3-stage Recursive Filter
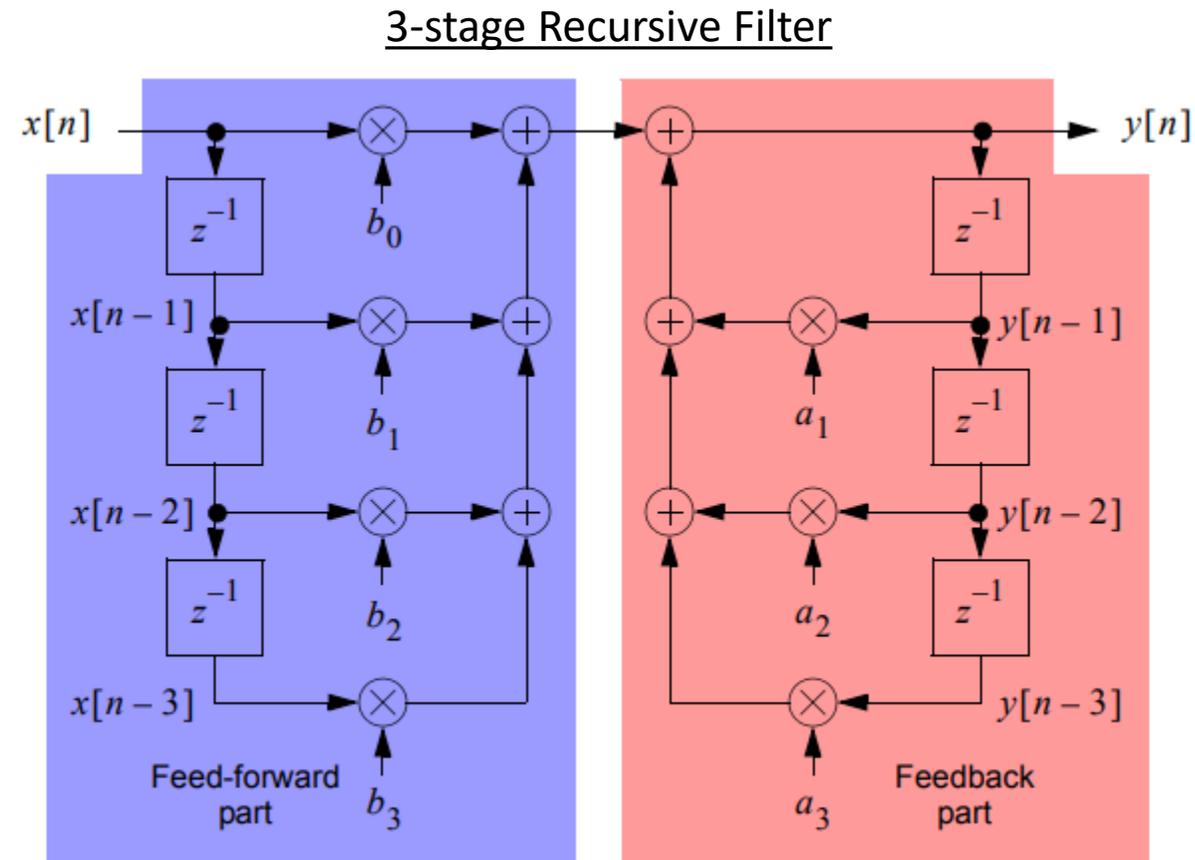
Single Pole Recursive Filter

LPF

HPF
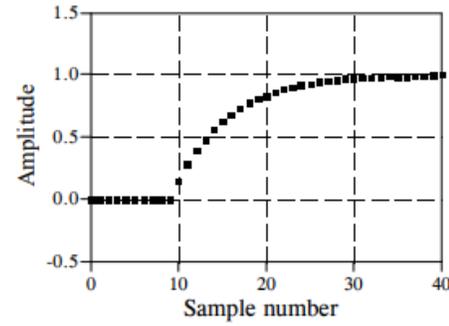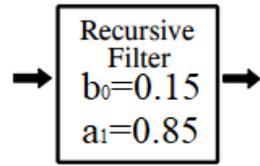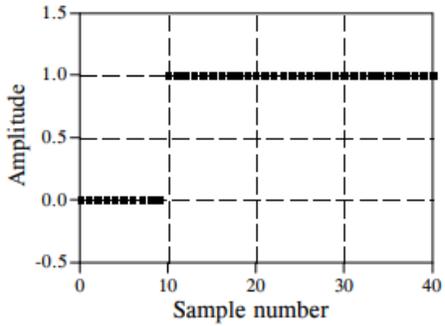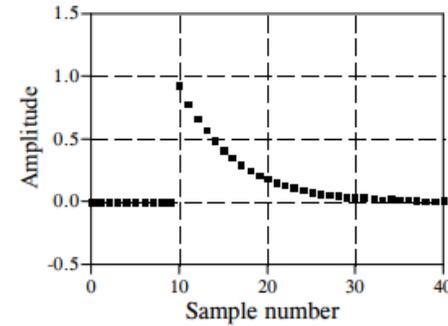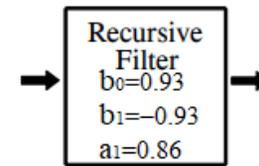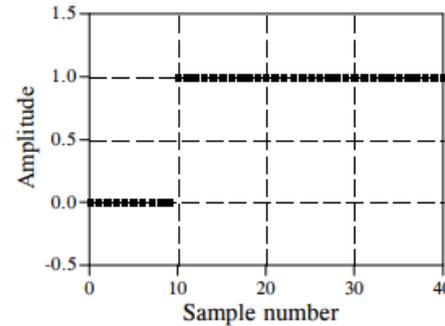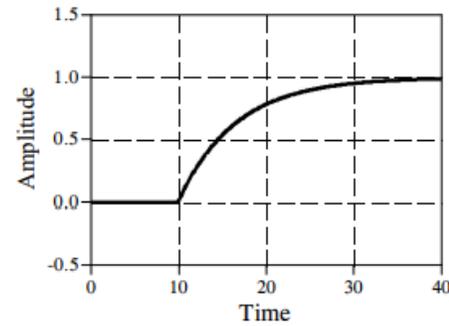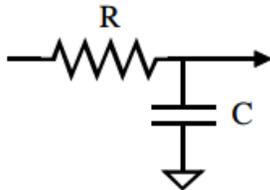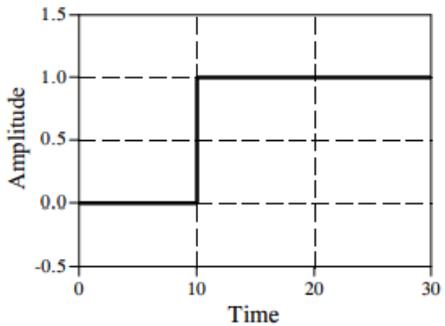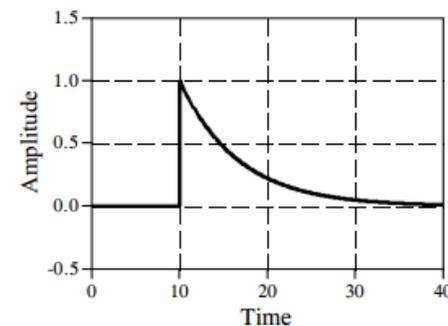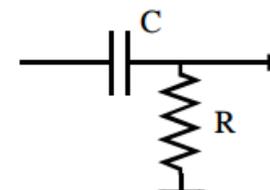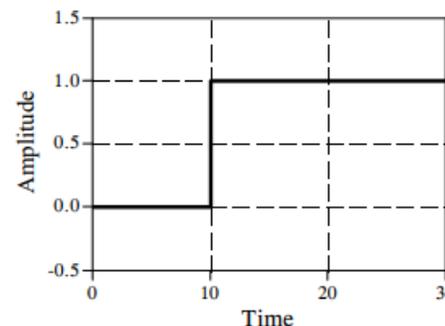
Proper *coefficient* selection can also make the **digital** recursive filter mimic an **analog** RC HPF or LPF.

Coefficient Selection:
(*Single Pole IIR Filter*)

➢LPF:

$b_0 = 1 - K$
$a_1 = K$

➢HPF:

$b_0 = \dfrac{1 + K}{2}$

$b_1 = -\dfrac{1 + K}{2}$

$a_1 = K$

- Physically, K is the amount of decay between adjacent samples.
- For example, $K = 0.87$ means that the value of each sample in the output signal is 0.87 the value of the sample before it.
- The higher the value of K, the slower the decay.
- $0 < K < 1$
- $K = e^{-2\pi f_C}$, where $f_C = -3$ dB cutoff frequency.

❑Single Pole Recursive Filter *in action*:

$K = 0.95$ (for LPF), $K = 0.86$ (for HPF)
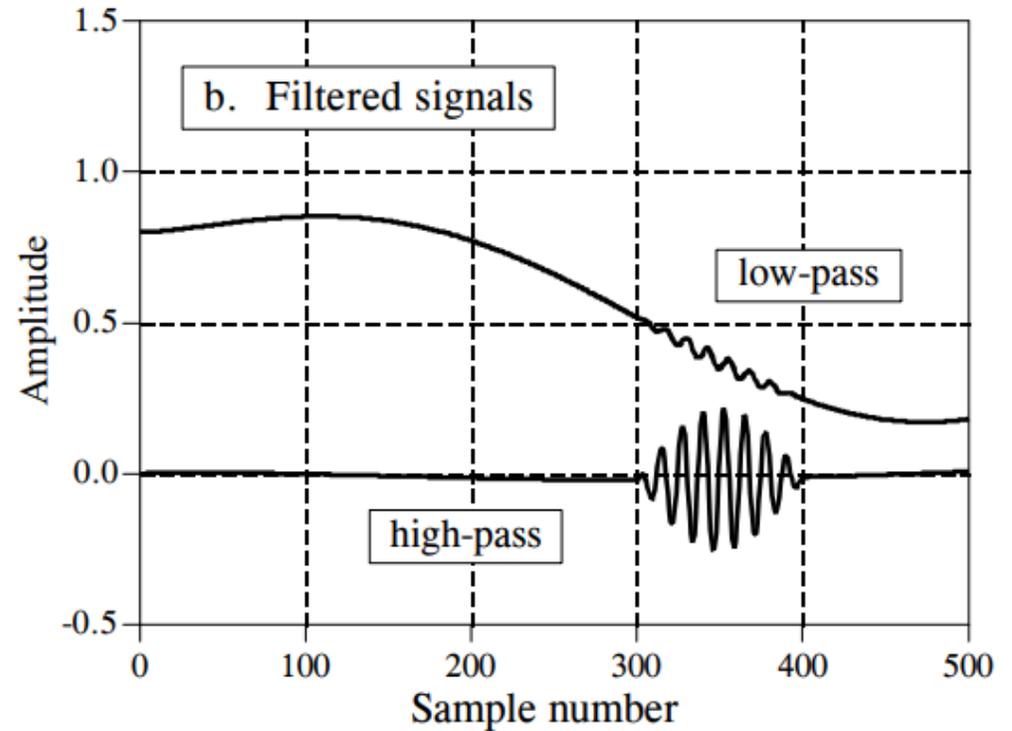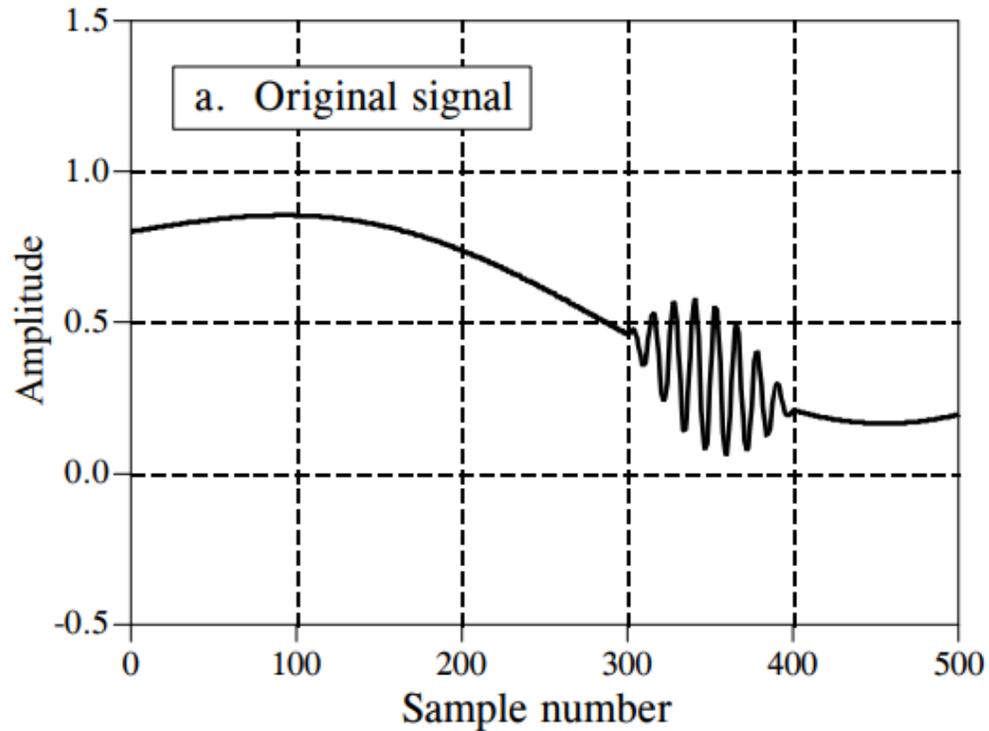
Remember: Single Pole Recursive Filter performs well in the time-domain, and poorly in the frequency-domain. Performance at higher f_C (with respect to sampling rate) is terrible!

Special: 4-stage Recursive LPF

[comparable to the Blackman, but faster]

Coefficient Selection:

$b_0 = (1 - K)^4$

$a_1 = 4K$

$a_2 = -6K^2$

$a_3 = 4K^3$

$a_4 = -K^4$

where $K = e^{-14.445f_c}$

Narrow-band Filters

Band-Pass Filter

$$b_0 = 1 - K_1$$
$$b_1 = 2(K_1 - K_2)\cos(2\pi f)$$
$$b_2 = K_2{}^2 - K_1$$
$$a_1 = 2K_2\cos(2\pi f)$$
$$a_2 = -K_2{}^2$$

Band-Reject Filter

$$b_0 = K_1$$
$$b_1 = -2K_1\cos(2\pi f)$$
$$b_2 = K_1$$
$$a_1 = 2K_2\cos(2\pi f)$$
$$a_2 = -K_2{}^2$$

where $K_2 = 1 - 3(BW)$;

$$K_1 = \frac{1 - 2K_2\cos(2\pi f) + K_2{}^2}{2 - 2\cos(2\pi f)}$$

a. Band-pass frequency response

single stage BW=0.0066

cascade of three stages
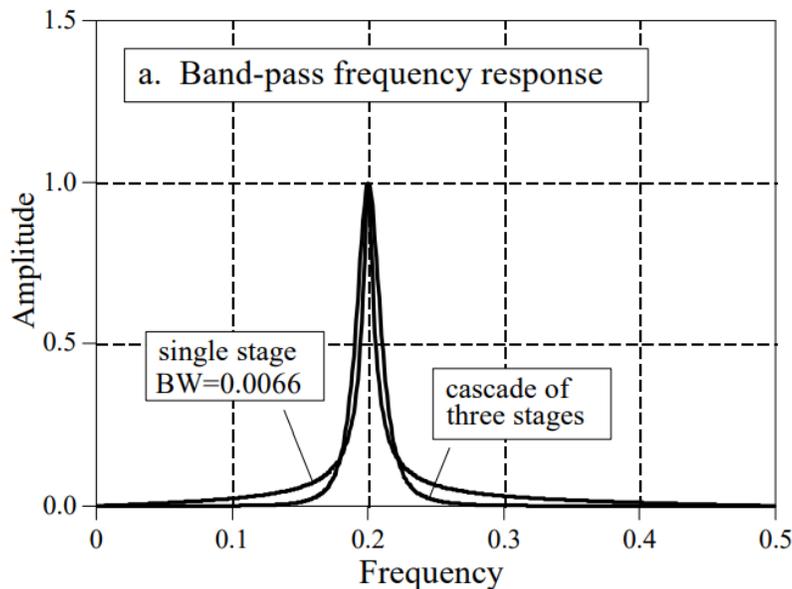
f =center frequency
BW =bandwidth measured
at 0.707 amplitude
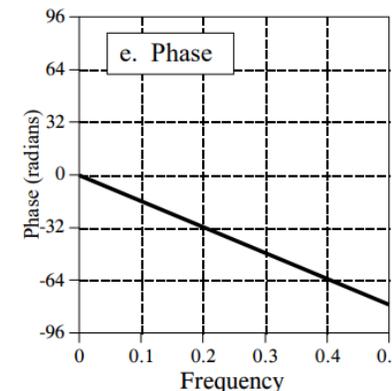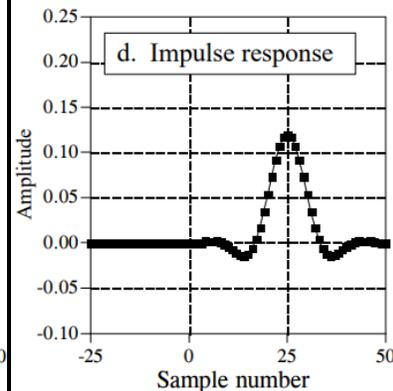[*both expressed as fraction of sampling frequency*]
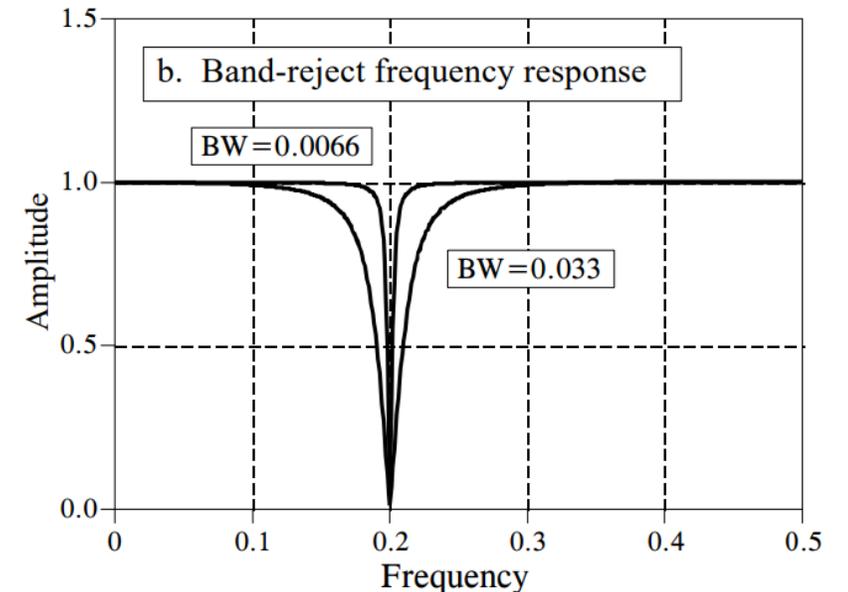
b. Band-reject frequency response

BW=0.0066

BW=0.033

Phase Response
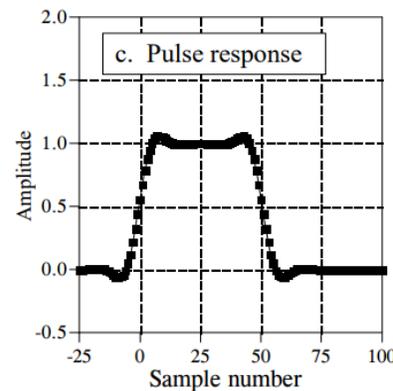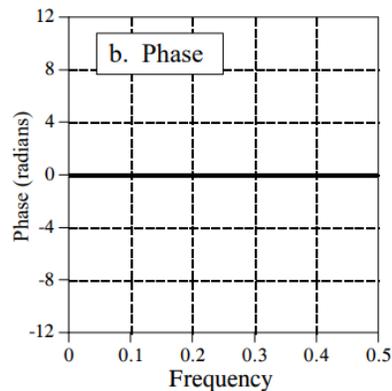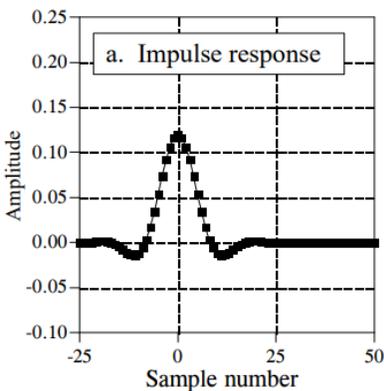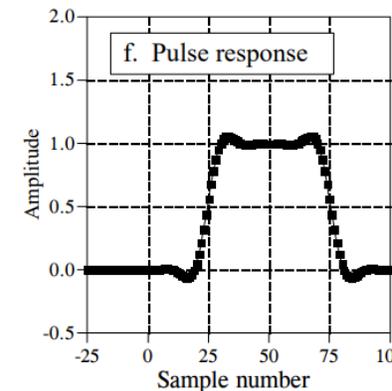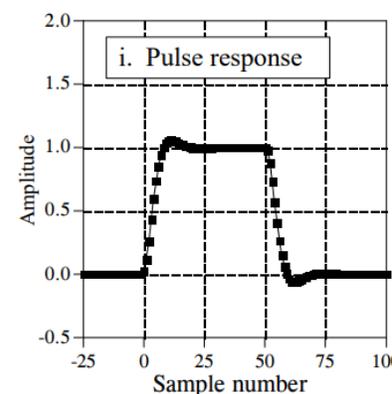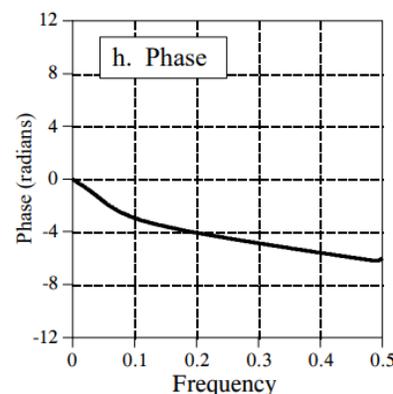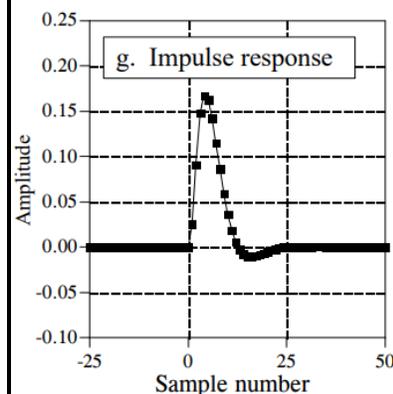
- What is so wrong with nonlinear phase? Many applications cannot tolerate the left and right edges looking different. This can be misinterpreted as a feature (terrible)!!!
- $h[n]$ of recursive filter is not symmetrical between left and right, therefore has a nonlinear phase.